Math 430 ①
Ch 10 Group Homomorphisms

Theorem 10.1.1. $\varphi(e) = \bar{e}$

Proof

$$e = e \cdot e$$

$$\varphi(e) = \varphi(e \cdot e) = \varphi(e)\varphi(e) \quad \text{Property 2}$$

$$\varphi(e) = \bar{e} \quad \text{Left cancellation.} \ \square$$

Theorem 10.1.2 $\varphi(g^n) = (\varphi(g))^n \ \forall n \in \mathbb{Z}$.

Proof Sketch

$n \in \mathbb{Z}^+$

$$\varphi(g^n) = \varphi(g^{n-1}g) = \varphi(g^{n-1})\varphi(g) \quad \text{Prop. 2.}$$

$$= \cdots = (\varphi(g))^n \quad \text{induction.}$$

$n \notin \mathbb{Z}^+$: same as Theorem 6.2 $\square$

Theorem 10.1.3 $|g| < \infty \Rightarrow |\varphi(g)| \mid |g|$.

Proof Let $|g| = n \in \mathbb{Z}^+$.

$$(\varphi(g))^n = \varphi(g^n) \quad \text{by Part 2.}$$

$$= \varphi(e) \quad \text{by } |g| = n$$

$$= \bar{e} \quad \text{by Part 1.}$$

$\therefore |\varphi(g)| \mid n$ by Cor 2 to Thm 4.1 $\square$

Theorem 10.1.4 $\operatorname{Ker}\varphi \leq G$.

Proof (1-step subgroup test)

nonempty: $\varphi(e) = \bar{e} \Rightarrow e \in \operatorname{Ker}\varphi$ (Part 1)

ab^{-1}: Let $a, b \in \operatorname{Ker}\varphi$.

$$\begin{aligned} \varphi(ab^{-1}) &= \varphi(a)\varphi(b^{-1}) && \text{Prop. 2} \\ &= \varphi(a)(\varphi(b))^{-1} && \text{Part 2} \\ &= \bar{e}\,\bar{e}^{-1} = \bar{e} && \text{by assumption.} \end{aligned}$$

and so $ab^{-1} \in \operatorname{Ker}\varphi$. $\square$

Theorem 10.1.5 $\varphi(a) = \varphi(b) \Leftrightarrow a\operatorname{Ker}\varphi = b\operatorname{Ker}\varphi$.

Proof ($\Rightarrow$) Let $a, b \in G$.

Assume $\varphi(a) = \varphi(b)$.

$$\begin{aligned} \bar{e} &= \varphi(b)(\varphi(a))^{-1} && \text{right cancellation} \\ &= \varphi(b)\varphi(a^{-1}) && \text{Part 2} \\ &= \varphi(ba^{-1}) && \text{Prop. 2} \end{aligned}$$

$\Rightarrow ba^{-1} \in \operatorname{Ker}\varphi$ ($ba^{-1} \in \operatorname{Ker}\varphi$ iff $ab^{-1} \in \operatorname{Ker}\varphi$)

$\Rightarrow b\operatorname{Ker}\varphi = a\operatorname{Ker}\varphi$ p138 part 4

Theorem 10.2.1-3

1. $H \leq G \Rightarrow \varphi(H) \leq \overline{G}$
2. H cyclic $\Rightarrow \varphi(H)$ cyclic.
3. H Abelian $\Rightarrow \varphi(H)$ Abelian.

Proof see Theorem 6.3. Relies only on property 2.

Theorem 10.2.4 $H \triangleleft G \Rightarrow \varphi(H) \triangleleft \varphi(G)$.

Proof (direct)

Assume $H \triangleleft G$.

Let $x \in \varphi(H)$ and $y \in \varphi(G)$.

Then there exist $h \in H$ and $g \in G$ with

$$\varphi(h) = x \quad \text{and} \quad \varphi(g) = y.$$

$$\begin{aligned} x y x^{-1} &= \varphi(h)\varphi(g)\varphi(h)^{-1} \\ &= \varphi(h g h^{-1}) \end{aligned}$$

$H \triangleleft G \Rightarrow h g h^{-1} \in H$.

Thus $\varphi(h g h^{-1}) \in \varphi(H)$.

By Normal Subgroup Test, $\varphi(H) \triangleleft \varphi(G)$.

$\square$

Theorem 10.2.8 $\bar{K} \triangleleft \bar{G} \Rightarrow \varphi^{-1}(\bar{K}) \triangleleft G$.

Proof By Normal Subgroup Test.

Let $x \in G$ and $k \in \varphi^{-1}(\bar{K})$.

Then $\varphi(k) \in \bar{K}$.

$$\varphi(xkx^{-1}) = \varphi(x)\varphi(k)(\varphi(x))^{-1}$$

and $\bar{K} \triangleleft \bar{G}$, so

$$\varphi(xkx^{-1}) \in \bar{K}.$$

But then $xkx^{-1} \in \varphi^{-1}(\bar{K})$.

Therefore $\varphi^{-1}(\bar{K}) \triangleleft G$. $\square$