Group Members: ____

Definition: Normal Subgroup
A subgroup H of a group G is called a *normal* subgroup if $aH = Ha$ for all $a \in G$. We denote this by $H \lhd G$.

Equivalently, $H \lhd G$ when the left cosets of H in G are *exactly the same* as the right cosets of H in G.

Theorem 9.1: Normal Subgroup Test
A subgroup H of G is normal in G iff $xHx^{-1} \subseteq H$ for all $x \in G$.

(1) Let $G = D_4$ and let $\langle R_{90} \rangle$ be the group of rotations of G.
(a) Write out the elements of $R_{90}\langle R_{90}\rangle R_{90}^{-1}$. Do you get $\langle R_{90}\rangle$ back?
(b) Write out the elements of $V\langle R_{90}\rangle V^{-1}$. Do you get $\langle R_{90}\rangle$ back?
(c) Write down the left cosets of $\langle R_{90}\rangle$ in G.
(d) Write out the right cosets of $\langle R_{90}\rangle$ in G.
(e) Inspect the Cayley Table of D_4 on p.33. View each left coset as a single element, collapsing rows and columns of the Cayley table, and write down the resulting new Cayley table.

(2) Let $G = \mathbb{Z}_8$ and let $H = \{0, 4\}$ be a subgroup of G.
(a) Write down the left cosets of H in G.
(b) Write down the right cosets of H in G.
(c) Alternatively, what does the fact that G is Abelian tell you about whether $H \lhd G$?
(d) Write the Cayley Table of G except with columns and rows grouped by coset of H in G. What group do you get if you view each coset as a single element?

(3) Refer to p.107 for this question about A_4, the rotations of the tetrahedron. Let $G = A_4$, and let $H = \{\alpha_1, \alpha_5, \alpha_9\}$. (Note that Table 5.1 is currently organized by the left cosets of $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ in G.)
(a) Write down the left cosets of H in G.
(b) Write down the right cosets of H in G.
(c) Reorder the Cayley table of A_4 by grouping elements by left coset of H in G.
(d) Is $H \lhd G$? Why or why not?

Theorem 9.2. Factor Groups
Let G be a group and let H be a normal subgroup of G. The set

$$G/H := \{aH \mid a \in G\}$$

is a group under the operation $(aH)(bH) = abH$. (This group is revealed by grouping the elements Cayley table of G by the cosets of a normal subgroup H of G.)

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(4) Recall that $2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$, under addition.
(a) Why is $2\mathbb{Z} \lhd \mathbb{Z}$?
(b) What are the cosets of $2\mathbb{Z}$ in $\mathbb{Z}$?
(c) Write the Cayley table for $\mathbb{Z}/2\mathbb{Z}$.

(5) Let $G = \mathbb{Z}_4 \oplus U(4)$, $H = \langle(2, 3)\rangle$, and $K = \langle(2, 1)\rangle$.
(a) Is $H \approx K$?
(b) Write the cosets and Cayley table for G/H.
(c) Write the cosets and Cayley table for G/K.
(d) Is $G/H \approx G/K$? (Expected, or a surprise?)