Group Members: ______________________________________________

**(1)** Give an example of a proper, nontrivial subgroup of $\mathbb{Z}_4 \oplus \mathbb{Z}_8$ that **cannot** be expressed in the form $H_1 \oplus H_2$, where $H_1 \leq \mathbb{Z}_4$ and $H_2 \leq \mathbb{Z}_8$.

---

**Theorem 8.2: Criterion for $G \oplus H$ to be Cyclic**
Let $G$ and $H$ be finite cyclic groups. Then $G \oplus H$ is cyclic iff $|G|$ and $|H|$ are relatively prime.

**Corollary 1: Criterion for $G_1 \oplus \cdots \oplus G_n$ to be Cyclic**
An external direct product $G_1 \oplus \cdots \oplus G_n$ of a finite number of finite cyclic groups is cyclic if and only if $|G_i|$ and $|G_j|$ are relatively prime when $i \neq j$.

**Corollary 2: Criterion for $\mathbb{Z}_{n_1 n_2 \cdots n_k} \approx \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$**
Let $m = n_1 n_2 \cdots n_k$. Then $\mathbb{Z}_m$ is isomorphic to $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ iff $n_i$ and $n_j$ are relatively prime when $i \neq j$.

---

**(2)** Is $\mathbb{Z}_3 \oplus \mathbb{Z}_9 \approx \mathbb{Z}_{27}$? Why or why not? Is $\mathbb{Z}_3 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{15}$? Why or why not?

**(3)** Use Corollary 2 and the fact that $G \oplus H \approx G \oplus K$ iff $H \approx K$ to determine whether $\mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 \approx \mathbb{Z}_{15} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12}$.

**Definition.** Let $n \geq 2$ be a positive integer, and let $k$ be a divisor of $n$. Then $U_k(n) = \{x \in U(n) \,|\, x \bmod k = 1\}$.

**Theorem 8.3:** $U(n)$ **as an External Direct Product**
Suppose $s$ and $t$ are relatively prime. Then

$$U(st) \approx U(s) \oplus U(t).$$

Moreover, $U_s(st) \approx U(t)$ and $U_t(st) \approx U(s)$.

**Proof Sketch:** The candidate isomorphism is $f : U(st) \to U(s) \oplus U(t)$ defined by

$$f(x) = (x \bmod s, x \bmod t). \tag{1}$$

For $U_s(st) \approx U(t)$, the candidate isomorphism is $g : U_s(st) \to U(t)$ defined by

$$g(x) = x \bmod t. \tag{2}$$

It remains to show that $f$ and $g$ are operation-preserving bijections.

**(4)** Write down the Cayley Table for $U(15)$.

**(5)** Theorem 8.3 states that $U(15) \approx U(3) \oplus U(5)$. Write out $f(x)$ in Equation (1) for all $x \in U(15)$.

**(6)** Theorem 8.3 states that $U_3(15) \approx U(5)$ and $U_5(15) \approx U(3)$. Just as in Question (2), write out $g(x)$ in Equation (2) for all $x$ in $U_3(15)$ and all $x \in U_5(15)$, respectively.

**(7)** Rewrite the Cayley table in Question (1) for $U(15)$, in the same order of columns and rows, except this time replacing the label $x \in U(15)$ with $f(x) = (x \bmod s, x \bmod t)$.

**(8)** Write down a group of the form $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ that is isomorphic to $U(15)$.

---

**Theorem: Structure of $U(n)$**
The groups $U(n)$ have the following structure. $U(2) = \{0\}$. $U(4) \approx \mathbb{Z}_2$.

$$\begin{aligned} U(2^n) &\approx \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{n-2}}, \quad \text{for } n \geq 3; \\ U(p^n) &\approx \mathbb{Z}_{p^n - p^{n-1}}, \quad \text{for } p \text{ an odd prime.} \end{aligned}$$

---

**(9)** Use the above theorems to write the following groups as the external direct products of cyclic groups: $U(10)$, $U(55)$, and $U(75)$.