Group Members: ____

Definition: External Direct Product
Let $G_1, G_2, \ldots, G_n$ be a finite collection of groups. The *external direct product* of $G_1, G_2, \ldots, G_n$, written as $G_1 \oplus G_2 \oplus \cdots \oplus G_n$, is the set of all n-tuples for which the ith component is an element of G_i, and the operation is componentwise:

$$(g_1, g_2, \ldots, g_n)(g_1', g_2', \ldots, g_n') = (g_1 g_1', g_2 g_2', \ldots, g_n g_n')$$

(notice the left-right order is preserved). Component-wise group operation means that in the ith entry, $g_i g_i'$ is computed in the group G_i.

(1) Write down the elements of $\mathbb{Z}_3 \oplus \mathbb{Z}_5$, using ellipses notation when the pattern is clear.

(2) Verify computationally that $\mathbb{Z}_3 \oplus \mathbb{Z}_5 = \langle (1,1) \rangle$.

Theorem 8.1: Order of an element in a direct product
The order of an element in a direct product of a finite number of finite groups is the least common multiple of the orders of the components of the element. In symbols,

$$|(g_1, \ldots, g_n)| = \operatorname{lcm}(|g_1|, \ldots, |g_n|).$$

Corollary of Theorem 4.4: Number of Elements of Order d in a Finite Group
In a finite group, the number of elements of order d is divisible by $\phi(d)$. (This is because the order d elements come from distinct cyclic subgroups of order d – these subgroups may overlap, but the overlap does not contain any order d elements. As a result you can compute the number of cyclic subgroups of order d by knowing the number of order d elements.)

(3) Determine the number of cyclic subgroups of order 15 in $\mathbb{Z}_{30} \oplus \mathbb{Z}_{20}$.

Definition: Direct Product Subgroup Notation
Let $G_1, \ldots, G_n$ be a finite number of groups, not necessarily all having finite order. For all $i = 1, \ldots, n$, let $H_i \leq G_i$. Then we define the set

$$H_1 \oplus \cdots \oplus H_n := \{(h_1, \ldots, h_n) \,|\, h_i \in H_i \text{ for all } i = 1, \ldots, n\},$$

which is a subgroup of $G_1, \ldots, G_n$ having order $|H_1 \oplus \cdots \oplus H_n| = |H_1| \cdot \cdots \cdot |H_n|$.

---

(4) Compute the order of $\langle 5 \rangle \oplus \langle 3 \rangle$ as a subgroup of $\mathbb{Z}_{30} \oplus \mathbb{Z}_{12}$. Is $\langle 5 \rangle \oplus \langle 3 \rangle$ cyclic? If so give a generator.

(5) Compute the order of $\langle 10 \rangle \oplus \langle 3 \rangle$ in $\mathbb{Z}_{30} \oplus \mathbb{Z}_{12}$. Is $\langle 10 \rangle \oplus \langle 3 \rangle$ cyclic? If so give a generator.

(6) Let G and H be groups. Prove that if G is not cyclic, then neither is $G \oplus H$. (Hint: try the contrapositive.)