Group Members: ____

Lagrange's Theorem: $|H|$ divides $|G|$
If G is a finite group and H is a subgroup of G, the $|H|$ divides $|G|$. Moreover, the number of distinct left (right) cosets of H in G is $|G|/|H|$.

Definition: Index of H in G
The *index* of H in G is written as $|G : H|$ and defined to be the number of cosets of H in G.

Corollary 1 $|G : H| = |G|/|H|$
If G is a finite group and H is a subgroup of G, then $|G : H| = |G|/|H|$.

Corollary 2 $|a|$ Divides $|G|$
In a finite group, the order of each element of the group divides the order of the group.

Corollary 3
A group of prime order is cyclic.

Corollary 4 $a^{|G|} = e$
Let G be a finite group, and let $a \in G$. Then $a^{|G|} = e$.

Corollary 5 Fermat's Little Theorem
For every integer a and every prime p, $a^p \mod p = a \mod p$.

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(1) Use what you know about cyclic groups to prove Corollary 2 of Lagrange's theorem.

(2) The following is a converse to Lagrange's Theorem:

> If d is a divisor of the order of a group G, then there exists a subgroup H of G with $|H| = d$.

Prove that A_4 is a counterexample to this converse as follows:
(a) Assume there does exist an order 6 subgroup $H \leq A_4$. Take any order 3 element $a \in A_4$ and look at the cosets H, aH, and a^2H. What does the pigeonhole principle say about H, aH and a^2H?
(b) Use the Page 139 Lemma to deduce that in all possible cases that $a \in H$.
(c) How many order 3 elements are there? What is the contradiction?

Theorem 7.2: Classification of Groups of Order $2p$
Let G be a group of order $2p$, where p is a prime greater than 2. Then G is isomorphic to Z_{2p} or D_p.

Definition: Stabilizer of a Point
Let G be a group of permutations of a set S. For each $i \in S$, let $\mathrm{stab}_G(i) = \{\phi \in G \mid \phi(i) = i\}$. We call $\mathrm{stab}_G(i)$ the *stabilizer of i in G*.

Definition: Orbit of a Point
Let G be a group of permutations of a set S. For each $s \in S$, let $\mathrm{orb}_G(s) = \{\phi(s) \mid \phi \in G\}$. The set $\mathrm{orb}_G(s)$ is called the *orbit of s under G*.

(3) Let $G = A_4$ be the group of even permutations on $\{1, 2, 3, 4\}$. Refer to p.107 to compute $\mathrm{stab}_G(1)$ and $\mathrm{orb}_G(1)$. What is the product of the sizes of these two sets?

(4) Let G be D_4, the set of plane symmetries of the square with side length 2 centered at the origin. Let p be the point with Cartesian coordinates $(\sqrt{2}/2, \sqrt{2}/2)$. Compute $\mathrm{stab}_G(p)$ and $\mathrm{orb}_G(p)$. What is the product of the sizes of these two sets?

(5) Let G be a group of permutations of a set S and let $i \in S$. Prove that $\mathrm{stab}_G(i)$ is a subgroup of G.

Theorem 7.3 Orbit-Stabilizer Theorem
Let G be a finite group of permutations of a set S. Then, for any i from S, $|G| = |\mathrm{orb}_G(i)||\mathrm{stab}_G(i)|$.

Theorem 7.4 The Rotation Group of a Cube
The group of rotations of a cube is isomorphic to S_4.