Group Members: ____

Definition: Coset of H in G
Let $a \in G$ and let H be a subgroup of a group G. The *left coset* of H in G containing a is

$$aH := \{ah \,|\, h \in H\},$$

and the *right coset* of H in G containing a is

$$Ha := \{ha \,|\, h \in H\}.$$

If a coset contains a, then a is a *coset representative* of that coset. For later use, define $aHa^{-1} = \{aha^{-1} \,|\, h \in H\}$.

Method for computing all cosets of H in G.
(I) Pick an element $a \in G$ that has not appeared yet in any coset.
(II) Determine/compute the coset containing this element a by group operation with h for all $h \in H$ (ah for left cosets, ha for right cosets).
(III) Stop when all elements of G appear in a coset. Write your answers in the form $aH = \ldots$, $bH = \ldots$, etc.

(1) Determine all of the left cosets and all of the right cosets of $2\mathbb{Z}$ (the even integers) in the integers $\mathbb{Z}$ (under addition). What do you notice?

(2) Refer to page 105 for this question. Let $G = A_4$ and let $H = \{\alpha_1, \alpha_5, \alpha_9\}$. Compute all of the left cosets of H in G in the following fashion:

(3) Repeat (1) except this time compute the right cosets of $\{\alpha_1, \alpha_5, \alpha_9\}$ in A_4.

Break. Proposition. If H is a subgroup of a group G, and $a \in Z(G)$, then $aH = Ha$.

Lemma: Properties of Cosets ("Page 139 Lemma").
Let H be a subgroup of a group G, and let $a, b \in G$. Then,
1. $a \in aH$,
2. $aH = H$ iff $a \in H$,
3. $aH = bH$ or $aH \cap bH = \emptyset$,
4. $aH = bH$ iff $a^{-1}b \in H$,
5. $|aH| = |bH|$,
6. $aH = Ha$ iff $H = aHa^{-1}$,
7. $aH \leq G$ iff $a \in H$.

(4) Why does the reverse direction ($a \in H \Rightarrow aH = H$) of the Lemma part 2 follow directly from the permutations constructed for the proof of Cayley's Theorem? (Hint: consider the mapping $T_a : H \to H$ defined by $T_a(h) = ah$.)

(5) Repeat the "Method for computing all cosets of H in G" on the top of the first page for Question (2), but *do not pick any of the elements a that you used before.* Compare the resulting cosets to the Lemma parts 3, 4, and 5.

(6) In questions (2-3), identify the cosets for which $aH = Ha$.

Left cosets partition G. From the Lemma we know that (i) the left cosets of H in G are the same size and nonempty, (ii) the union of the left cosets of H in G is G, and (iii) distinct left cosets are *pairwise disjoint*. Therefore the set of left cosets of G is a *partition* of G (see p.17). When G is finite there is a finite list $a_1H, \ldots, a_rH$ of all distinct left cosets, and

$$\begin{aligned} \sum_{i=1}^{r} |a_iH| &= |G| \qquad \text{(all } a \in G \text{ appear in some coset)} \\ r \cdot |H| &= |G| \qquad \text{(all cosets have the same size),} \end{aligned}$$

and so the order of a subgroup H divides the order of the group G when $|G|$ is finite! This is Lagrange's Theorem.
Note: we could have stated the previous paragraph in terms of right cosets.