Group Members: ______________________________

(1) Referring to the Cayley Table for D_3 on the back cover, identify the elements of D_3 with the set $\{1, 2, 3, 4, 5, 6\}$, and write down the permutation in S_6 corresponding to each row of D_3.

Break.

Theorem 6.1 Cayley's Theorem
Every group is isomorphic to a group of permutations.

---

(2) Label the corners of an equilateral triangle with $\{1, 2, 3\}$ (increasing in the counter-clockwise direction), and design a group isomorphism $\phi : D_3 \to S_3$ by righting down how each $g \in D_3$ permutes $\{1, 2, 3\}$. (Why are (1) and (2) *both* relevant to Cayley's Theorem?)

Theorem 6.2 Properties of Isomorphism Acting on Elements
Suppose that ϕ is an isomorphism from a group G onto a group $\overline{G}$. Then
1. ϕ carries the identity of G to the identity of $\overline{G}$.
2. For every $n \in \mathbb{Z}$ and every $a \in G$, $\phi(a^n) = [\phi(a)]^n$.
3. For any elements $a, b \in G$, a and b commute iff $\phi(a)$ and $\phi(b)$ commute.
4. $G = \langle a \rangle$ iff $\overline{G} = \langle \phi(a) \rangle$.
5. $|a| = |\phi(a)|$ for all $a \in G$ (isomorphisms preserve orders).
6. For a fixed integer k and a fixed $b \in G$, the equation $x^k = b$ has the same number of solutions in G as does the equation $x^k = \phi(b)$ in $\overline{G}$.
7. If G is finite, then G and $\overline{G}$ have exactly the same number of elements of every order.

Theorem 6.3 Properties of Isomorphism Acting on Groups.
Suppose that ϕ is an isomorphism from a group G onto a group $\overline{G}$. Then
1. G is Abelian iff $\overline{G}$ is Abelian.
2. G is cyclic iff $\overline{G}$ is cyclic.
3. ϕ^{-1} is an isomorphism from $\overline{G}$ onto G.
4. If K is a subgroup of G, then $\phi(K) = \{\phi(k) \mid k \in K\}$ is a subgroup of $\overline{G}$.

---

(3) Describe how to apply Theorem 6.2 to prove Theorem 6.3 part 1. Technical details are not necessary.

(4) Describe how to apply Theorem 6.2 to prove Theorem 6.3 part 2. Technical details are not necessary.

(5) We know that if ϕ is a bijection, then so is ϕ^{-1}. Prove that ϕ^{-1} is operation preserving in order to prove Theorem 6.3 part 4.

(6) Use a subgroup test to prove Theorem 6.3 part 4.

Definition: Automorphism.
An isomorphism from a group G to itself is called an *automorphism* of G.

Definition: Inner Automorphism Induced by $a \in G$.
Let G be a group, and let $a \in G$. The function ϕ_a defined by $\phi_a(x) = axa^{-1}$ for all x in G is called the inner automorphism of G induced by a.

(7) Recall that Theorem 6.2 part 5 says that an isomorphism preserves the order of an element. That means for an automorphism ϕ of a cyclic group $\mathbb{Z}_n$, $\phi(1)$ must be a generator of $\mathbb{Z}_n$. Fill out the table with the distinct automorphisms of $\mathbb{Z}_8$ by specifying for each automorphism where each element is mapped.

$\mathbb{Z}_8$	1	2	3	4	5	6	7	0
ϕ_a								
ϕ_b								
ϕ_c								
ϕ_d								

(8) Now compute the compositions $\phi_b \circ \phi_c$ and $\phi_c \circ \phi_d$. What are the resulting ϕ's?

$\mathbb{Z}_8$	1	2	3	4	5	6	7	0	Result
$\phi_b \circ \phi_c$									
$\phi_b \circ \phi_d$									

(9) Assuming that the ϕ's above form a group by composition, complete the Cayley table for the ϕ's. What group have you seen before that is isomorphic to this group?

$\mathbb{Z}_8$	ϕ_a	ϕ_b	ϕ_c	ϕ_d
ϕ_a				
ϕ_b				
ϕ_c				
ϕ_d				

Break.

Theorem 6.4 Aut(G) **and** Inn(G) **are Groups**
The set of automorphisms of a group and the set of inner automorphisms of a group are both groups under the operation of function composition.

Theorem 6.5 $\text{Aut}(\mathbb{Z}_n) \approx U(n)$
For every $n \in \mathbb{Z}^+$, $\text{Aut}(\mathbb{Z}_n) \approx U(n)$.

(8) Prove that if G is Abelian, then there is a unique inner automorphism of G (i.e., $|\text{Inn}(G)| = 1$).

Visualization of $\phi_{\alpha_5} \in \text{Inn}(A_4)$
Note that $g \mapsto \alpha_5 g \alpha_5^{-1}$ under ϕ_{α_5}. On the tetrahedron, the first rotation applied is α_5^{-1}, which changes the axes of rotation of the tetrahedron. Then $\alpha_5 g \alpha_5^{-1}$ is the same *kind* of rotation as g, but with respect to the *new labels* that appear at the position of the *original labels* corresponding to g.

(9) Refer to Table 5.1 on page 107 for this question. Compute the inner automorphisms ϕ_{α_1} and ϕ_{α_5} by tabulating the image of each permutation in A_4 under ϕ_{α_5}. Do this by looking up entries in Table 5.1 and without actually computing products of permutations. (Replace the original labels of $g \in A_4$ with the corresponding new labels appearing in the original positions after ϕ^{-1} is applied.)

A_4	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
ϕ_{α_1}												
ϕ_{α_5}												

For thought. Visualize the inner automorphism ϕ_{α_5} by practicing the rotations on the tetrahedron.