Group Members: ____

Break.

Cycle Notation for Permutations.
I. Graphical representation of cycle structure
II. Cycles operating as functions on elements
III. (Right-associative) composition of cycles

Theorem 5.1. Every permutation can be written as a product of disjoint cycles. (By the deterministic algorithm described in Group Activity 5A)

(1) Consider two disjoint cycles, $\sigma = (1\,2\,4)$ and $\tau = (3\,5\,7)$. Convert both $\sigma\tau$ and $\tau\sigma$ into two-line notation. Do the same thing for $\alpha = (2\,4\,5)$ and $\beta = (3\,4\,5)$. What do you notice?

(2a) Compute the order of $(1\,2\,3\,4\,5\,6)$.
(2b) Compute the order of $\sigma\tau$ from (1).
(2c) Compute the order of $(1\,2\,4)(3\,5)$.

Break.

Theorem 5.2 Disjoint Cycles Commute. If the pair of cycles $\alpha = (a_1\,a_2\,\ldots\,a_m)$ and $\beta = (b_1\,b_2\,\ldots\,b_n)$ have no entries in common, then $\alpha\beta = \beta\alpha$.

Theorem 5.3 Order of a Permutation. The order of a permutation is the least common multiple of the lengths of the cycles in disjoint cycle form.

(3) Write the permutation $(1\,5)(1\,4)(1\,3)(1\,2)$ as a product of disjoint cycles following the discussion of cycle notation in I–III above. Write the permutation $(a_1\,a_m)(a_1\,a_{m-1})\cdots(a_1\,a_3)(a_1\,a_2)$ as the product of disjoint cycles. Is this process always reversible? Set up the skeleton of a proof by induction, showing the base case and the inductive hypothesis.

Break.

Theorem 5.4 Product of 2-Cycles. Every permutation in S_n, $n > 1$, is a product of 2-cycles.

Proof sketch. Step 1. Use Thm. 5.1 to write the permutation in disjoint cycle notation.
Step 2. Convert each cycle to a product of 2-cycles as in (3).

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(4) Define ε to be the empty cycle, that is, the identity permutation in S_n. The list of *equal* permutation pairs below are *rewrite rules* used to convert between all possible representations of ε as products of 2-cycles. Show by exhaustive mapping of $\{a, b, c, d\}$ that the last two pairs in the list are valid rewrite rules.

$(a\,b)$ and $(b\,a)$
ε and $(a\,b)(a\,b)$
$(a\,b)(b\,c)$ and $(a\,c)(a\,b)$
$(a\,c)(c\,b)$ and $(b\,c)(a\,b)$
$(a\,b)(c\,d)$ and $(c\,d)(a\,b)$

(5) Use the rewrite rules of (4) to reduce the following permutation to the identity element ε:
$(1\,4)(2\,3)(1\,2)(1\,4)(2\,4)(2\,3)$

Break.

Lemma. If $\varepsilon = \beta_1\beta_2\cdots\beta_r$, where the β's are 2-cycles, then r is even.

Proof sketch. Let $i \in \{1, \ldots, n\}$ be the smallest number in any of the 2-cycles. Use the rewrite rules to move strictly decrease the rightmost occurrence of i. Eventually i must cancel via an operation $(ij)(ij) = \varepsilon$; otherwise only the leftmost 2-cycle contains i, and the permutation is not the identity.

Break.

Theorem 5.5. Always even or always odd
If a permutation α can be expressed as a product of an even (odd) number of 2-cycles, then every decomposition of α into a product of 2-cycles must have an even (odd) number of 2-cycles. In symbols, if

$$\alpha = \beta_1\beta_2\cdots\beta_4 \quad \text{and} \quad \alpha = \gamma_1\gamma_2\cdots\gamma_s,$$

where the β's and the γ's are 2-cycles, then r and s are both even or both odd.

Theorem 5.6 Even Permutations form a Group
The set of even permutations in S_n (called the alternating group A_n) forms a subgroup of S_n.

Proof. A straightforward exercise using properties discussed above.

Theorem 5.7. For $n > 1$, A_n has order $n!/2$.

Proof sketch. Prove that the function $f : A_n \to S_n \setminus A_n$ defined by $f(\alpha) = (12)\alpha$ is a bijection.