

Group Members: _____

Definition. The *order* of a group G , denoted by $|G|$, is the number of elements in G . Either $|G| = \infty$, or $|G|$ is a positive integer; a finite group is sometimes indicated by $|G| < \infty$.

Definition. The *order* of an element g of a group G , denoted $|g|$, is the smallest positive integer n such that $g^n = e$ ($n \cdot g = 0$ in additive notation). If no such integer exists, we say g has *infinite order* and write $|g| = \infty$.

(1) Classify the elements of the groups $U(8)$ and $U(10)$ according to their orders (see Group Activity 2A Problem (6)).

(2) Classify the elements of the group \mathbb{R}^* under multiplication according to their orders.

(3) Classify the elements of the group \mathbb{Z}_{12} under addition mod n according to their orders.

(4) Classify the elements of the group \mathbb{Z} under addition according to their orders.

Definition. If a subset H of a group G is itself a group under the operation of G , we say that H is a subgroup of G .

Remark. If H is a subset of some G that we already know is a group, we have a head start on proving that H itself is a group.

What we know already: (i) the candidate binary operation on H is the one on G , (ii) the candidate binary operation on H is associative by inheritance.

What we must show about H : (i) the candidate binary operation is **closed** on H , (ii) H has an identity (the identity of G), and (iii) H contains inverses of all of its elements.

(5) List 5 subgroups of the nonzero complex numbers \mathbb{C}^* under multiplication.

(6) By inspecting the Cayley tables of $U(8)$ and $U(10)$, list all of the subgroups of $U(8)$ and $U(10)$.

$U(8)$:

$U(10)$:

(7) By inspecting the Cayley table of D_3 , list all of its subgroups. Visualize the result of restricting to certain rows and columns. (Hint: there are 6 subgroups.)

D_3	R_0	R_{120}	R_{240}	F_1	F_2	F_3
R_0	R_0	R_{120}	R_{240}	F_1	F_2	F_3
R_{120}	R_{120}	R_{240}	R_0	F_3	F_1	F_2
R_{240}	R_{240}	R_0	R_{120}	F_2	F_3	F_1
F_1	F_1	F_2	F_3	R_0	R_{120}	R_{240}
F_2	F_2	F_3	F_1	R_{240}	R_0	R_{120}
F_3	F_3	F_1	F_2	R_{120}	R_{240}	R_0

Break. Theorem 3.1 One-Step Subgroup Test. Let G be a group and H a nonempty subset of G . If ab^{-1} is in H whenever a and b are in H , then H is a subgroup of G . (In additive notation, if $a - b$ is in H whenever a and b are in H , then H is a subgroup of G .)

Usage. 1. Identify the defining condition for H . 2. Prove the identity e of G fulfills this condition. 3. Assume some a, b in G fulfill the condition. 4. Prove that for this a, b that ab^{-1} fulfills the condition.

(8) Use the One-step subgroup test to prove that the even integers are a subgroup of \mathbb{Z} under addition.

Break. Theorem 3.2 Two-Step Subgroup Test. Let G be a group and let H be a nonempty subset of G . If ab is in H whenever a and b are in H (binary operation closure), and a^{-1} is in H whenever a is in H (closure of inverses), then H is a subgroup of G .

Usage. 1. Identify the defining condition for H . 2. Prove the identity e of G fulfills this condition. 3. Assume some a, b in G fulfill the condition. 4. Prove for this a, b that ab and a^{-1} fulfill the condition.

(9) Use the Two-step subgroup test to prove that the subset H of an Abelian group G defined by

$$H = \{g \in G : |g| \leq 2\},$$

that is, the subset of elements with order at most 2, is a subgroup of G .

Break. Theorem 3.3 Finite Subgroup Test. Let H be a nonempty finite subset of a group G . If H is closed under the operation of G , then H is a subgroup of G .

Usage. 1. Identify the defining condition for H . 2. Prove some $a \in G$, such as e , fulfills this condition. 3. Assume some a, b in G fulfill the condition. 4. Prove for this a, b that ab fulfills the condition.

(10) Use the finite subgroup test to prove the following. Let G be a finite group and let $a \in G$. Then the set $H = \{a^i \mid i = 1, 2, 3, \dots\}$ is a subgroup of G .

(11) Prove that if a is an element of a group G , then the order of a is less than or equal to the order of G (Hint: get the easy case of $|G| = \infty$ out of the way first.)