Group Members: ____

(1) Without referring to the book, prove Theorem 10.1 Part 4: $\text{Ker}\,\phi$ is a subgroup of G when $\phi : G \to \overline{G}$ is a homomorphism. Use the 1-step or 2-step subgroup test. (Thm. 10.1 Part 2 is helpful.)

(2) Prove the reverse implication in Part 5 of Theorem 10.1. That is, $a\text{Ker}\,\phi = b\text{Ker}\,\phi \Rightarrow \phi(a) = \phi(b)$ for any homomorphism $\phi : G \to \overline{G}$. (Use Part 5 of the p.139 Lemma.)

(3) Without referring to the book, prove Theorem 10.2 Part 7, that is, if $\overline{K} \leq \overline{G}$, then $\phi^{-1}(\overline{K}) \leq G$. (Part 8 states that "$\leq$" can be replaced by "$\lhd$" in both places.)

(4) Define the homomorphism $\phi : D_4 \to \mathbb{Z}_{10}$ by sending all rotations to 0 and letting $\phi(H) = 5$.
(a) Compute $D_4/\text{Ker}\,\phi$ and $\phi(D_4)$.
(b) Are $D_4/\text{Ker}\,\phi$ and $\phi(D_4)$ the same size? Isomorphic? If so give an isomorphism.

(5) Define the homomorphism $\phi : D_4 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by $\phi(R_0) = \phi(R_{180}) = (0, 0)$, $\phi(R_{90}) = (1, 0)$, and $\phi(D) = (0, 1)$.
(a) Finish the definition of ϕ for all elements of D_4.
(b) Compute $D_4/\text{Ker}\,\phi$ and $\phi(D_4)$.
(c) Are $D_4/\text{Ker}\,\phi$ and $\phi(D_4)$ the same size? Isomorphic? If so give an isomorphism.

Theorem 10.3: First Isomorphism Theorem
Let $\phi : G \to \overline{G}$ be a group homomorphism. Then the mapping from $G/\text{Ker}\,\phi$ to $\phi(G)$, given by $g\text{Ker}\,\phi \to \phi(g)$, is an isomorphism, so that $G/\text{Ker}\,\phi \approx \phi(G)$.

Corollary
If $\phi : G \to \overline{G}$ is a homomorphism, then $|\phi(G)|$ divides both $|G|$ and $|\overline{G}|$.

(6) For every normal subgroup N of $\mathbb{Z}_{15}$, define a homomorphism $\phi : \mathbb{Z}_{15} \to \mathbb{Z}_{15}$ with $\operatorname{Ker} \phi = N$. (See Exercise 41 for a more general statement about the number of homomorphisms from $\mathbb{Z}_n$ to $\mathbb{Z}_k$.)

Theorem 10.4: Normals are Kernel
Every normal subgroup of a group G is the kernel of a homomorphism of G. In particular, a normal subgroup N is the kernel of the mapping $g \to gN$ from G to G/N.

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(7) For at least 5 normal subgroups N of D_6 (plane symmetries of the hexagon), define a homomorphism $\phi : D_6 \to \overline{G}$ with $\operatorname{Ker} \phi = N$. Write $\overline{G}$ in a commonly used form such as $\mathbb{Z}_2$ or $\{e\}$ rather than D_6/N. (Hint: $\{R_0, R_{120}, R_{240}\} \lhd D_6$.)