Theorem 9.1 Normal subgroup Test

A subgroup H of G is normal in G iff $xHx^{-1} \subseteq H$ for all x in G.

Proof

($\Rightarrow$) Assume $H \triangleleft G$.

Let $x \in G$.

$$xHx^{-1} = Hxx^{-1} \quad \text{since } H \triangleleft G$$
$$= He = H.$$

($\Leftarrow$) Assume $xHx^{-1} \subseteq H$ for all $x \in G$.

Let $a \in G$.

Set $x = a$ to obtain $aHa^{-1} \subseteq H$

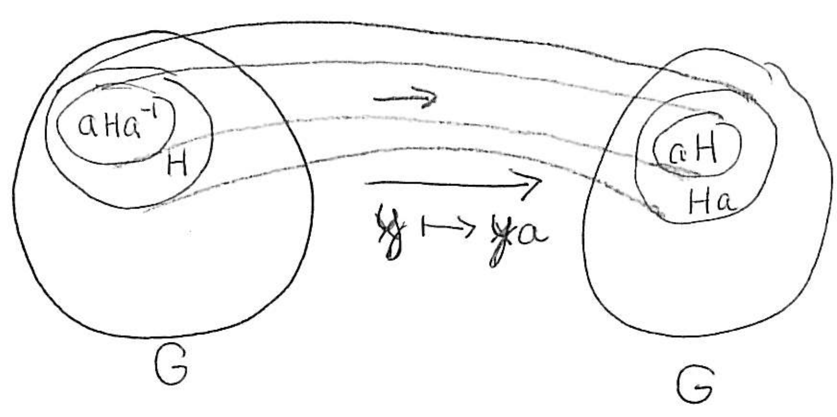
or $aH \subseteq Ha$.

Set $x = a^{-1}$ to obtain $a^{-1}Ha \subseteq H$

or $Ha \subseteq aH$.

Taken together, $aH = Ha$.

Therefore $H \triangleleft G$. $\square$



Theorem 9.2 Factor Groups

Let G be a group and let H be a normal subgroup of G. The set $G/H = \{aH \mid a \in G\}$ is a group under the operation $(aH)(bH) = abH$.

Proof

The key is well-definedness. Does

$$aH = a'H \text{ and } bH = b'H \Rightarrow (aH)(bH) = (a'Hb'H)?$$

Suppose $aH = a'H$ and $bH = b'H$.

Then $a'a^{-1} \in H$ and $b'b^{-1} \in H$ by p138 part 4.

Then $a' = ah_1$ and $b' = bh_2$ for some $h_1, h_2 \in H$.

Computing:

$$
\begin{aligned}
(a'H)(b'H) &= a'b'H && \text{by Defn.}\\
&= ah_1bh_2H && \text{by lemma p138 part 4}\\
&= ah_1bH && \text{by lemma p138 part 2}\\
&= ah_1Hb && \text{since } H \triangleleft G\\
&= aHb && \text{by lemma p138 part 2}\\
&= abH && \text{since } H \triangleleft G\\
&= (aH)(bH) && \text{by definition.}
\end{aligned}
$$

Thus the operation is a well-defined, closed binary operation.

identity eH

inverses $(aH)^{-1} = a^{-1}H$

associativity $((aH)(bH))(cH) = (ab)cH = a(bc)H$
$= (aH)((bH)(cH))$ $\square$

Thm 9.3 Let G be a group. If $G/Z(G)$ is cyclic, then G is Abelian.

Proof

Since $G/Z(G)$ is cyclic, there is a coset $gZ(G)$, for some $g \in G$, such that

$$G/Z(G) = \langle gZ(G) \rangle = \{Z(G), gZ(G), g^{-1}Z(G), \ldots\}$$

Now to show G is Abelian, let $a, b \in G$.

$aZ(G), bZ(G) \in G/Z(G)$ implies that

$$aZ(G) = g^i Z(G) \text{ for some } i \in \mathbb{Z}.$$
$$bZ(G) = g^j Z(G) \text{ for some } j \in \mathbb{Z}.$$

Therefore $a = g^i x$ for some $x \in Z(G)$
$b = g^j y$ for some $y \in Z(G)$,
by p. 138 part 4.

Computing,
$$\begin{aligned} ab &= g^i x g^j y \\ &= g^i g^j x y \quad \text{since } x \in Z(G) \\ &= g^{i+j} y x \quad \text{since } x \in Z(G) \\ &= g^j g^i y x = g^j y g^i x \\ &= ba. \end{aligned}$$

Since a, b were arbitrary, G is Abelian. $\square$

Thm 9.4 If G is a group, then $G/Z(G) \approx \mathrm{Inn}(G)$.

Proof

We require an operation preserving bijection

$$T: G/Z(G) \to \mathrm{Inn}(G).$$

Since $G/Z(G) = \{ gZ(G) \mid g \in G \}$, and

$\mathrm{Inn}(G) = \{ \varphi_g \mid g \in G \}$ where $\varphi_g(x) = gxg^{-1}$ $\forall x \in G$,

We try the easiest possibility:

$$T: \{ gZ(G) \mid g \in G \} \to \{ \varphi_g \mid g \in G \}$$
$$T(gZ(G)) = \varphi_g$$

T is well-defined. Let $g_1 Z(G) = g_2 Z(G)$.

By p. 138 part 4, $g_1 = g_2 z$ for $z \in Z(G)$.

Let $x \in G$. Then

$$\varphi_{g_1}(x) = g_1 x g_1^{-1} = g_2 z x (g_2 z)^{-1}$$
$$= g_2 z x z^{-1} g_2^{-1} = g_2 x z z^{-1} g_2^{-1}$$
$$= g_2 x g_2^{-1} = \varphi_{g_2}(x),$$

and so $\varphi_{g_1} = \varphi_{g_2}$ and T is well defined.

Thm 9.4 $G/Z(G) \approx \mathrm{Inn}(G)$.

$(\; T(g(Z(G))) = \varphi_g \;)$

T is 1-1

Let $g_1, g_2 \in G$ and suppose $T(g_1 Z(G)) = T(g_2(Z_G))$;
i.e., that $\varphi_{g_1} = \varphi_{g_2}$.
Then for all $x \in G$, $\varphi_{g_1}(x) = \varphi_{g_2}(x)$, or

$$g_1 x g_1^{-1} = g_2 x g_2^{-1}$$
$$g_2^{-1} g_1 x g_1^{-1} = x g_2^{-1}$$
$$g_2^{-1} g_1 x = x g_2^{-1} g_1$$

Therefore $g_2^{-1} g_1 \in Z(G)$, and by p138 part 4,
$$g_1 Z(G) = g_2 \; Z(G).$$

T is onto Let $\varphi_g \in \mathrm{Inn}(G)$.
Then $g \in G$ and $gZ(G) \in G/Z(G)$,
with $T(gZ(G)) = \varphi_g$.

T operation preserving

Let $g_1 Z(G), g_2 Z(G) \in G/Z(G)$.
$T(g_1 Z(G) g_2 Z(G)) = T(g_1 g_2 Z(G)) = \varphi_{g_1 g_2}$.
$\forall x \in G$, $\varphi_{g_1 g_2}(x) = g_1 g_2 x (g_1 g_2)^{-1} = g_1 g_2 x g_2^{-1} g_1^{-1}$
$= g_1 (g_2 x g_2^{-1}) g_1^{-1} = g_1 \varphi_{g_2}(x) g_1^{-1}$
$= \varphi_{g_1}(\varphi_{g_2}(x))$.

Thus $\varphi_{g_1 g_2} = \varphi_{g_1} \varphi_{g_2} = T(g_1 Z(G)) T(g_2 Z(G))$. $\square$