Theorem 7.3 Orbit - Stabilizer Theorem

Let G be a finite group of permutations of a set S. Then for any $i \in S$,

$$|G| = |orb_G(i)| \cdot |stab_G(i)|.$$

Proof Let

$\varphi_1 stab_G(i), \varphi_2 stab_G(i), \ldots, \varphi_r stab_G(i)$

be the distinct left cosets of $stab_G(i)$ in G.

By Lagrange's Theorem, $r = |G : stab_G(i)| = |G|/|stab_G(i)|$.

We construct and prove a bijection

$$T : \{\varphi_j stab_G(i) \mid j = \{1, \ldots, r\}\} \to orb_G(i).$$

Define $T(\varphi\, stab_G(i)) = \varphi(i)$.

Claim 1 T is well-defined.

(We need this because there are many choices of φ to produce the same left coset $\varphi\, stab_G(i)$).

Suppose $\varphi_x stab_G(i) = \varphi_y stab_G(i)$.

By Lemma p 138 #4, $\varphi_x^{-1}\varphi_y \in stab_G(i)$.

By definition, this means $\varphi_x^{-1}\varphi_y(i) = i$

$$\varphi_x^{-1}\varphi_y(i) = \varphi_x^{-1}(\varphi_y(i)) = i$$

iff $\varphi_x(i) = \varphi_y(i)$ by defn. inverse.

Therefore $\varphi_x stab_G(i) = \varphi_y stab_G(i) \Rightarrow$

$$T(\varphi_x stab_G(i)) = \varphi_x(i) = \varphi_y(i) = T(\varphi_y stab_G(i))$$

and T is well-defined.

Claim 2 T is 1-1.

Let $\varphi_x \text{Stab}_G(i), \varphi_y \text{stab}_G(i)$ be two left cosets.

Assume $T(\varphi_x \text{stab}_G(i)) = T(\varphi_y \text{stab}_G(i))$.

By definition of T, $\varphi_x(i) = \varphi_y(i)$.

φ_x is a permutation, and so has inverse φ_x^{-1}.

By definition of inverse function,

$$\varphi_x(i) = \varphi_y(i) \text{ iff } \varphi_x^{-1}(\varphi_y(i)) = i.$$

$$\varphi_x^{-1}\varphi_y(i) = \varphi_x^{-1}(\varphi_y(i)) = i \Rightarrow \varphi_x^{-1}\varphi_y \in \text{stab}_G(i).$$

By lemma p 138 part 4, $\varphi_x \text{stab}_G(i) = \varphi_y \text{stab}_G(i)$

and T is 1-1.

Claim 3 T is onto.

Let $j \in \text{orb}_G(i)$.

Then there exists some $\varphi \in G$ with $\varphi(i) = j$.

Simply notice $T(\varphi \text{stab}_G(i)) = \varphi(i) = j$. $\square$

Rotations of a tetrahedron

Label the sides 1, 2, 3, 4. $S = \{1, 2, 3, 4\}$

How many permutations stabilize 1?

ε, (234), (243)

What is the orbit of 1?

$\{1, 2, 3, 4\}$

$$|\text{stab}_G(1)| \cdot |\text{orb}_G(1)| = 3 \cdot 4 = 12 = |A_4| \checkmark$$

Corollary There are — rotations of the cube.

The set of faces of the cube is $S = \{1,2,3,4,5,6\}$

(we can think of these as the points in the center of the respective faces.)

Set $i = 1$. Let G be the rotations of the cube.

$Stab_G(i) = 4$ since the axis through i and the opposite point has $0°, 90°, 180°$, and $270°$ rotations which stabilize i.

Clearly $orb_G(i) = \{1, 2, 3, 4, 5, 6\}$.

Therefore $|G| = 4 \cdot 6 = 24$.

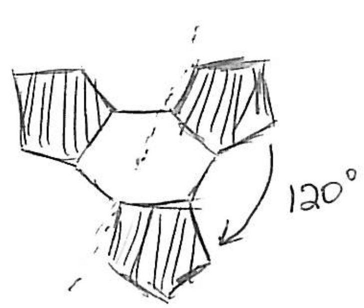
Corollary There are — rotations of the truncated icosohedron (soccer ball)

There are 20 hexagons and 12 pentagons on the truncated icosohedron. Let G = rotations.

Let $i \in S = \{\text{hexagons}\}$

$j \in T = \{\text{pentagons}\}$

		$stab_G(i)$	$orb_G(i)$	product
hex.	i	3 (??)	20	60
pent.	j	5	12	60



The 60° rotation does not send pentagons to pentagons.