Theorem 7.3 Orbit-Stabilizer Theorem

Let \( G \) be a finite group of permutations of a set \( S \). Then for any \( i \in S \),

\[ |G_i| = |\text{orb}_G(i)| \cdot |\text{stab}_G(i)|. \]

Proof: Let

\( \mathcal{C}_1, \text{stab}_G(i), \mathcal{C}_2, \text{stab}_G(i), \ldots, \mathcal{C}_r, \text{stab}_G(i) \)

be the distinct left cosets of \( \text{stab}_G(i) \) in \( G \).

By Lagrange's Theorem, \( r = |G : \text{stab}_G(i)| = |G|/|\text{stab}_G(i)|. \)

We construct and prove a bijection

\[ T : \{\mathcal{C}_j \cap \text{stab}_G(i) \mid j = 1, \ldots, r\} \to \text{orb}_G(i). \]

Define \( T(\mathcal{C}_j \cap \text{stab}_G(i)) = \mathcal{C}_j(i). \)

Claim 1: \( T \) is well-defined.

(We need this because there are many choices of \( \mathcal{C}_j \) to produce the same left coset \( \mathcal{C}_j \cap \text{stab}_G(i) \).)

Suppose \( \mathcal{C}_x \cap \text{stab}_G(i) = \mathcal{C}_y \cap \text{stab}_G(i) \).

By Lemma p138 #4, \( \mathcal{C}_x^{-1} \mathcal{C}_y \in \text{stab}_G(i) \).

By definition, this means \( \mathcal{C}_x^{-1} \mathcal{C}_y(i) = i \)

\[ \mathcal{C}_x^{-1} \mathcal{C}_y(i) = \mathcal{C}_x^{-1} (\mathcal{C}_y(i)) = i \]

iff \( \mathcal{C}_x(i) = \mathcal{C}_y(i) \) by defn. inverse.

Thus \( \mathcal{C}_x \cap \text{stab}_G(i) = \mathcal{C}_y \cap \text{stab}_G(i) \) \implies \( T(\mathcal{C}_x \cap \text{stab}_G(i)) = \mathcal{C}_x(i) = \mathcal{C}_y(i) = T(\mathcal{C}_y \cap \text{stab}_G(i)) \)

and \( T \) is well-defined.
Claim 2. \( T \) is 1-1.

Let \( (\ell x \, \text{stab}_G(i)), (\ell y \, \text{stab}_G(i)) \) be two left cosets.

Assume \( T((\ell x \, \text{stab}_G(i))) = T((\ell y \, \text{stab}_G(i))) \).

By definition of \( T \), \( (\ell x(i)) = (\ell y(i)) \).

\( \ell x \) is a permutation, and so has inverse \( \ell x^{-1} \).

By definition of inverse function,

\[ \ell x(i) = \ell y(i) \iff \ell x^{-1}(\ell y(i)) = i, \]

\[ \ell x^{-1}\ell y(i) = (\ell x^{-1}(\ell y(i))) = i \Rightarrow \ell x^{-1}\ell y \in \text{stab}_G(i). \]

By Lemma \( \rho \) 138 pent 4, \( \ell x \, \text{stab}_G(i) = \ell y \, \text{stab}_G(i) \) and \( T \) is 1-1.

Claim 3. \( T \) is onto.

Let \( j \in \text{orb}_G(i) \).

Then there exists some \( \ell \in G \) with \( \ell(i) = j \).

Simply notice \( T((\ell \, \text{stab}_G(i))) = \ell(i) = j. \)

Rotations of a tetrahedron

Label the sides 1, 2, 3, 4. \( S = \{1, 2, 3, 4\} \).

How many permutations stabilize 1?

\( \varepsilon, (234), (243) \)

What is the orbit of 1?

\( \varepsilon, 1, 2, 3, 4 \)

\( |\text{stab}_G(1)| \cdot |\text{orb}_G(1)| = 3 \cdot 4 = 12 = |A_4| \)

\( \checkmark \)
Corollary: There are rotations of the cube.

The set of faces of the cube is \( S = 1, 2, 3, 4, 5, 6 \) (we can think of these as the points in the center of the respective faces.)

Set \( i = 1 \). Let \( G \) be the rotations of the cube. \( \text{Stab}_G(i) = 4 \) since the axis through \( i \) and the opposite point has \( 0, 90, 180, \) and \( 270^\circ \) rotations which stabilize \( i \).

Clearly \( \text{Orb}_G(i) = 1, 2, 3, 4, 5, 6 \).

Therefore \( |G| = 4 \cdot 6 = 24 \).

Corollary: There are rotations of the truncated icosahedron (soccer ball).

There are 20 hexagons and 12 pentagons on the truncated icosahedron. Let \( G = \text{rotations} \).

Let \( i \in S = \| \text{hexagons} \| \)

\( j \in T = \| \text{pentagons} \| \)

<table>
<thead>
<tr>
<th></th>
<th>( \text{Stab}_G(i) )</th>
<th>( \text{Orb}_G(i) )</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>hex. i</td>
<td>3 (??)</td>
<td>20</td>
<td>60</td>
</tr>
<tr>
<td>pent. j</td>
<td>5</td>
<td>12</td>
<td>60</td>
</tr>
</tbody>
</table>

The 60° rotation does not send pentagons to pentagons.