Theorem 6.1 Cayley’s Theorem

Every group is isomorphic to a group of permutations.

Proof

Let $G$ be any group.

We need to construct a group $\overline{G}$ of permutations, with an isomorphism $\varphi : G \to \overline{G}$.

Q. Permutations on what? A. on $G$.

Idea

$$
g \leftrightarrow T_g
$$

Element of $G$ \hspace{1cm} Permutation on $G$

Definition For each $g \in G$, define

$$
T_g : G \to G \hspace{1cm} \text{by}
$$

$$
T_g(x) = g \cdot x \hspace{1cm} \text{for all } x \in G.
$$

Fact 1 $T_g$ is a permutation on $G$ (Ex. 21)

Fact 2 Set $\overline{G} = \{ T_g | g \in G \}$.

Then $\overline{G}$ is a group under function composition.
Fact 2  Proof

We claim \( T_g \circ T_h = T_{gh} \), so that function composition is a closed binary operation.

\[
T_g \circ T_h (x) = T_g (h \cdot x) = gh \cdot x = (gh)x = T_{gh} (x).
\]

\( gh \in G \Rightarrow T_{gh} \in G \), so it's closed.

associativity  Yes, b/c function composition.

identity  \( T_e \circ T_g (x) = T_g (x) = T_{ge} (x) = T_g (x) = T_g T_e (x) \); so

\( T_e \in G \) is the identity in \( G \).

inverses  \( T_g \circ T_{g^{-1}} (x) = T_{gg^{-1}} (x) = T_e (x) \)

\[
= T_{g^{-1}g} (x) = T_{g^{-1}} T_g (x),
\]

so \( (T_g)^{-1} = T_{g^{-1}} \) is the inverse of \( T_g \).

Isomorphism  Define \( \Phi : G \to \overline{G} \)

by \( \Phi (g) = T_g \).

Fact 3  \( \Phi \) is a bijection (exercise)

Fact 4  \( \Phi \) is operation preserving

\[ \Phi (xy) = T_{xy} = T_x T_y = \Phi (x) \Phi (y) \]  \( \square \)