Theorem 5.2 Disjoint cycles Commute

Proof sketch:

Let $\alpha = (a_1\ a_2 \cdots a_\ell)$ be disjoint
$\beta = (b_1\ b_2 \cdots b_m)$ cycles in S_n

Let $C = \{1, \dots, n\} \sim (\{a_1, \dots, a_\ell\} \cup \{b_1, \dots, b_m\})$

We check that $\alpha\beta(x) = \beta\alpha(x)$ for all $x \in$ ~~S_n~~ $\{1, \dots, n\}$

Case 1 $x \in \{a_1, \dots, a_\ell\}$

Thus $x = a_i$ for some $1 \le i \le \ell$. cyclic behavior!

Then $\alpha\beta(a_i) = \alpha(a_i) = a_{(i \bmod \ell)+1}$

$= \beta(a_{(i \bmod \ell)+1})$

$= \beta\alpha(a_i)$.

Case 2 $x \in \{b_1, \dots, b_m\}$ by switching roles of α and β in Case 1.

Case 3 $x \in C$.

Then $\alpha\beta(x) = \alpha(x)$ since $x \notin \{b_1, \dots, b_m\}$

$= x$ since $x \notin \{a_1, \dots, a_\ell\}$

$= \beta(x) = \beta(\alpha(x))$

$= \beta\alpha(x)$

All cases $x \in \{1, \dots, n\}$ are covered. $\square$

Theorem 5.3 The order $|\alpha\beta|$ of disjoint cycles α, β in S_N is the least common multiple of the lengths of α and β.

Proof: Let α have length m, and β length n.

Claim 1 $|\alpha| = m$ and $|\beta| = n$ (Exercise!)

$$\boxed{\begin{array}{l}\text{Set } k = \operatorname{lcm}(m,n). \\ \text{Set } t = |\alpha\beta|.\end{array}}$$

Claim 2 t divides k

$$(\alpha\beta)^k = \alpha^k\beta^k \qquad \text{since } \alpha\beta = \beta\alpha \text{ (Thm 5.2)}$$
$$= \varepsilon \cdot \varepsilon \qquad \text{since } m|k,\ n|k$$

Therefore $t|k$ by Theorem 4.1 (Cor 2).

Claim 3 k divides t

$$(\alpha\beta)^t = \alpha^t\beta^t = \varepsilon, \text{ since } |\alpha\beta| = t$$

Thus $\alpha^t = \beta^{-t}$.

But α, β are disjoint, so the only possibility is $\alpha^t = \beta^{-t} = \varepsilon$.

Therefore $|\alpha|\,|\,t$ and $|\beta|\,|\,t$ by Thm 4.1,

In other words, $m|t$ and $n|t$, so

k divides $|\alpha\beta| = t$.

By Claims 2 and 3, $|\alpha\beta| = \operatorname{lcm}(|\alpha|, |\beta|)$.

Remark We extend to ≥ 3 cycles by induction and the property

$$\operatorname{lcm}(l, m, n) = \operatorname{lcm}(l, \operatorname{lcm}(m,n)). \quad \square$$

Theorem 5.5 Always even or always odd

A permutation $\alpha \in S_n$ is either even or odd. This means whenever α is written as a product of 2-cycles

$$\alpha = \beta_1 \beta_2 \cdots \beta_r,$$

that α even $\Leftrightarrow$ r is even

α odd $\Leftrightarrow$ r is odd.

Proof Suppose β_i's + γ_j's are 2-cycles with

$$\alpha = \beta_1 \beta_2 \cdots \beta_r = \gamma_1 \gamma_2 \cdots \gamma_s.$$

Then $\varepsilon = \beta_1 \beta_2 \cdots \beta_r (\gamma_1 \gamma_2 \cdots \gamma_s)^{-1}$

$$= \beta_1 \cdots \beta_r \gamma_1^{-1} \cdots \gamma_s^{-1}$$

where the γ_j^{-1} are 2 cycles.

By the lemma, $r+s$ is even. Therefore r and s have the same parity.

By transitivity, all ways of writing α as the product of 2-cycles have the same parity. ∎