Question. When are two groups the “same”?

Definition. An isomorphism $\phi$ from a group $G$ to a group $\overline{G}$ is a bijection from $G$ to $\overline{G}$ that preserves the operation. That is,

$$\phi(ab) = \phi(a)\phi(b) \quad \text{for all } a, b \text{ in } G.$$ 

If there is an isomorphism from $G$ onto $\overline{G}$, we say that $G$ and $\overline{G}$ are isomorphic and write $G \equiv \overline{G}$.

(1) Suppose we were to write down the Cayley tables of two isomorphic groups $G$ and $\overline{G}$. How can an isomorphism from $G$ to $\overline{G}$ be described in terms of the Cayley tables?

For problems (2)-(4) exhaustively describe the isomorphisms for these small groups.

(2) Find an isomorphism from $\{−1, 1\}$ under multiplication to $\mathbb{Z}_2$.

(3) Find two distinct isomorphisms from the cyclic subgroup of rotations in $D_3$ to $\mathbb{Z}_3$.

(4) Let $G$ be the group $\{(0,0), (1,0), (0,1), (1,1)\}$ under coordinate-wise addition mod 2. Find an isomorphism between $G$ and the group generated by the 180 degree rotations of the tetrahedron. Is this isomorphism unique?

General Procedure for Proving Isomorphism

Step 1. Define the candidate mapping $\phi$ from $G$ to $\overline{G}$.

Step 2. Prove that $\phi$ is one-to-one.

Step 3. Prove that $\phi$ is onto.

Step 2. Prove that for all $a, b \in G$, $\phi(ab) = \phi(a)\phi(b)$.
Definition. An automorphism is an isomorphism from a group to itself.

(5) Prove that $\phi(x) = \sqrt{x}$ is an automorphism on $\mathbb{R}^+$, the group of positive real numbers under multiplication.

(6) Prove that $U(8)$ is not isomorphic to $U(10)$.

(7) Prove that $S_4$ is not isomorphic to $D_{12}$. 