Break.

Cycle Notation.
I. Graphical interpretation of cycle structure
II. Cycles operating as functions on elements
III. (Right-associative) composition of cycles

Facts about Cycles.

Theorem 5.1. Every permutation can be written as a product of disjoint cycles. (By the deterministic algorithm described in Group Activity 5A)

(1) Consider two disjoint cycles, $\sigma = (1\,2\,4)$ and $\tau = (3\,5\,7)$. Convert both $\sigma\tau$ and $\tau\sigma$ into two-line notation. Do the same thing for $\alpha = (2\,4\,5)$ and $\beta = (3\,4\,5)$. What do you notice?

(2a) Compute the order of $(1\,2\,3\,4\,5\,6)$.
(2b) Compute the order of $\sigma\tau$ from (1).
(2c) Compute the order of $(1\,2\,4)(3\,5)$.

Break.

Theorem 5.2 Disjoint Cycles Commute. If the pair of cycles $\alpha = (a_1\,a_2\,\ldots\,a_m)$ and $\beta = (b_1\,b_2\,\ldots\,b_n)$ have no entries in common, then $\alpha\beta = \beta\alpha$.

Theorem 5.3 Order of a Permutation. The order of a permutation is the least common multiple of the lengths of the cycles in disjoint cycle form.

(3) Write the permutation $(1\,5)(1\,4)(1\,3)(1\,2)$ as a product of disjoint cycles following the discussion of cycle notation in I–III above. Write the permutation $(a_1\,a_m)(a_1\,a_{m-1})\cdots(a_1\,a_3)(a_1\,a_2)$ as the product of disjoint cycles. Is this process always reversible? Prove it by induction.

(Continued on Reverse)

Break.

Theorem 5.4 Product of 2-Cycles. Every permutation in S_n, $n > 1$, is a product of 2-cycles.

(4) Define ε to be the empty cycle, that is, the identity permutation in S_n. By considering the operation of the following cycles on $\{a, b, c, d\}$, show that each pair is the same permutation. This gives us a set of *rewrite rules* for permutations written as products of 2-cycles.

$(a\,b)$ and $(b\,a)$
ε and $(a\,b)(a\,b)$
$(a\,b)(b\,c)$ and $(a\,c)(a\,b)$
$(a\,c)(c\,b)$ and $(b\,c)(a\,b)$
$(a\,b)(c\,d)$ and $(c\,d)(a\,b)$

(5) Use the rewrite rules of (4) to reduce the following permutation to the identity element ε:
$(1\,4)(2\,3)(1\,2)(1\,4)(2\,4)(2\,3)$

Break.

Lemma. If $\varepsilon = \beta_1\beta_2 \cdots \beta_r$, where the β's are 2-cycles, then r is even.