Definition. The order of a group $G$, denoted by $|G|$, is the number of elements in $G$. Either $|G| = \infty$, or $|G|$ is a positive integer; a finite group is sometimes indicated by $|G| < \infty$.

Definition. The order of an element $g$ of a group $G$, denoted $|g|$, is the smallest positive integer $n$ such that $g^n = e$ ($n \cdot g = 0$ in additive notation). If no such integer exists, we say $g$ has infinite order and write $|g| = \infty$.

(1) Classify the elements of the groups $U(8)$ and $U(10)$ according to their orders (see Group Activity 2A Problem (6)).

(2) Classify the elements of the group $\mathbb{R}^*$ under multiplication according to their orders.

(3) Classify the elements of the group $\mathbb{Z}_{12}$ under addition mod $n$ according to their orders.

(4) Classify the elements of the group $\mathbb{Z}$ under addition according to their orders.

Definition. If a subset $H$ of a group $G$ is itself a group under the operation of $G$, we say that $H$ is a subgroup of $G$.

If $H$ is a subset of some $G$ that we already know is a group, we have a head start on proving that $H$ itself is a group.

What we know already: (i) the candidate binary operation on $H$ is the one on $G$, (ii) the candidate binary operation on $H$ is associative by inheritance.

What we must show about $H$: (i) the candidate binary operation is closed on $H$, (ii) $H$ has an identity (the identity of $G$), and (iii) $H$ contains inverses of all of its elements.

(Continued on reverse)
(5) List 5 subgroups of the nonzero complex numbers $\mathbb{C}^*$ under multiplication.

(6) By inspecting the Cayley tables of $U(8)$ and $U(10)$, list all of the subgroups of $U(8)$ and $U(10)$.

$U(8)$:

$U(10)$:

(7) By inspecting the Cayley table of $D_3$, list all of its subgroups. Visualize the result of restricting to certain rows and columns. (Hint: there are 6 subgroups.)

$$
\begin{array}{c|cccc}
D_3 & R_0 & R_{120} & R_{240} & F_1 & F_2 & F_3 \\
R_0 & R_0 & R_{120} & R_{240} & F_1 & F_2 & F_3 \\
R_{120} & R_{120} & R_0 & R_{240} & F_3 & F_1 & F_2 \\
R_{240} & R_{240} & R_{120} & R_0 & F_2 & F_3 & F_1 \\
F_1 & F_1 & F_2 & F_3 & R_0 & R_{120} & R_{240} \\
F_2 & F_2 & F_3 & F_1 & R_{240} & R_0 & R_{120} \\
F_3 & F_3 & F_1 & F_2 & R_{120} & R_{240} & R_0 \\
\end{array}
$$

Break. **Theorem 3.1 One-Step Subgroup Test.** Let $G$ be a group and $H$ a nonempty subset of $G$. If $ab^{-1}$ is in $H$ whenever $a$ and $b$ are in $H$, then $H$ is a subgroup of $G$. (In additive notation, if $a - b$ is in $H$ whenever $a$ and $b$ are in $H$, then $H$ is a subgroup of $G$.)

**Usage.** 1. Identify the defining condition for $H$. 2. Prove the identity $e$ of $G$ fulfills this condition. 3. Assume some $a, b$ in $G$ fulfill the condition. 4. Prove that for this $a, b$ that $ab^{-1}$ fulfills the condition.

(8) Use the One-step subgroup test to prove that the even integers are a subgroup of $\mathbb{Z}$ under addition.

(9) Use the One-step subgroup test to prove that the subset $H$ of an Abelian group $G$ defined by

$$
H = \{g \in G : |g| \leq 2\},
$$

that is, the subset of elements with order at most 2, is a subgroup of $G$. 