Group Members: ____

Isomorphisms Versus Homomorphisms

Let $\phi : G \to \overline{G}$ be a function, where G and $\overline{G}$ are groups.

Property 1. ϕ is a bijection.
Property 2. ϕ is operation-preserving; i.e., $\forall x, y \in G$, $\phi(xy) = \phi(x)\phi(y)$.

Definition. ϕ is a *group isomorphism* if it has Properties 1&2, and we say $G \approx \overline{G}$.

Definition. ϕ is a *group homomorphism* if it has Property 2.

Definition. The *kernel* of a homomorphism $\phi : G \to \overline{G}$ is the set

$$\mathrm{Ker}\phi = \phi^{-1}(\overline{e}) = \{x \in G \mid \phi(x) = \overline{e}\},$$

where $\overline{e}$ is the identity element of $\overline{G}$. The kernel is the set of elements of G that map to the identity element of $\overline{G}$. This is the *preimage* of $\overline{e}$ under ϕ.

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(1) Define $\phi : \mathbb{Z} \to \mathbb{Z}_2$ by

$$\phi(x) = \begin{cases} 0 & \text{if } x \text{ is even,} \\ 1 & \text{if } x \text{ is odd.} \end{cases}$$

(a) Does ϕ have Property 1? If so prove it. **(b)** Does ϕ have Property 2? If so prove it.
(c) What is the kernel of ϕ? **(d)** What are $\phi^{-1}(1)$ and $\phi^{-1}(1)$?

(2) Define $G = \{ax + b \mid a, b \in \mathbb{R}\}$ to be the set of degree 0 and 1 polynomials in the variable x over the real numbers. Define the function $\phi : G \to G$ by

$$\phi(ax + b) = \frac{d}{dx}(ax + b).$$

(a) Does ϕ have Property 1? If so prove it. **(b)** Does ϕ have Property 2? If so prove it.
(c) What is the kernel of ϕ? **(d)** What are $\phi^{-1}(0)$, $\phi^{-1}(5)$, and $\phi^{-1}(x)$?

(3) Let $n \in \mathbb{Z}^+$. Recall that $\mathbb{R}^n$ can be viewed as the set of $n \times n$ column vectors over the real numbers. Let M be an $n \times n$ matrix with real coefficients, and define $\phi : \mathbb{R}^n \to \mathbb{R}^n$ to be the linear transformation

$$\phi(\mathbf{x}) = M\mathbf{x}.$$

Use what you know about matrices to answer the following.
(a) Under what condition does ϕ have Property 1?
(b) Does ϕ have Property 2? If so prove it.
(c) What is the matrix theory/linear algebra name for the kernel of ϕ?
(d) When does the equation $M\mathbf{x} = \mathbf{b}$ have exactly one solution? More than one solution? No solutions? Try to use group theory language.

Theorem 10.1: Properties of Homomorphisms
Let $\phi : G \to \overline{G}$ be a group homomorphism. Let G have identity e and $\overline{G}$ have identity $\overline{e}$. Then
1. $\phi(e) = \overline{e}$.
2. $\phi(g^n) = (\phi(g))^n$ for all $n \in \mathbb{Z}$.
3. If $|g|$ is finite, then $|\phi(g)|$ divides $|g|$.
4. Kerϕ is a subgroup of G.
5. $\phi(a) = \phi(b)$ iff aKer$\phi = b$Kerϕ.
6. If $\phi(g) = g'$, then $\phi^{-1}(g') = \{x \in G \mid \phi(x) = g'\} = g\text{Ker}\phi$.

Theorem 10.2: Properties of Subgroups Under Homomorphisms
Let $\phi : G \to \overline{G}$ be a group homomorphism, and let $H \leq G$. Let G have identity e and $\overline{G}$ have identity $\overline{e}$. Then
1. $\phi(H) = \{\phi(h) \mid h \in H\}$ is a subgroup of $\overline{G}$.
2. If H is cyclic, then $\phi(H)$ is cyclic.
3. If H is Abelian, then $\phi(H)$ is Abelian.
4. If $H \lhd G$, then $\phi(H) \lhd \overline{G}$.
5 If $|\text{Ker}\phi| = n$, then ϕ is an n-to-1 mapping from G onto $\phi(G)$.
6. If $|H| = n$, then $|\phi(H)|$ divides n.
7. If $\overline{K} \leq \overline{G}$, then $\phi^{-1}(\overline{K}) = \{k \in G \mid \phi(k) \in \overline{K}\} \leq G$.
8. If $\overline{K} \lhd \overline{G}$, then $\phi^{-1}(\overline{K}) = \{k \in G \mid \phi(k) \in \overline{K}\} \lhd G$.
9. If ϕ is onto and Ker$\phi = \{e\}$, then ϕ is an isomorphism from G to $\overline{G}$.

Kernels are Normal: Set $K = \{\overline{e}\}$ in Property 8 to see that $\text{Ker}\phi = \phi^{-1}(\overline{e}) \lhd G$.

Theorem 10.3: First Isomorphism Theorem.
Let $\phi : G \to \overline{G}$ be a group homomorphism. Then $G/\text{Ker}\phi \approx \phi(G)$, under the isomorphism $g\text{Ker}phi \to \phi(g)$.
