I. Examples, Counterexamples and short answer. (2 pts ea.) Do not give proofs, but clearly indicate your proposed example or counterexample, or short answer where appropriate.

1. Clearly circle the letter(s) of the transformation(s) below that are plane symmetries of a square.

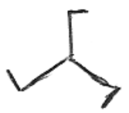
   (a) Reflection about the vertical axis of the square.

   (b) Translation to the right by 1 unit.

   (c) Contraction by half under the mapping $(x, y) \mapsto (x/2, y/2)$.

   (d) Rotation clockwise by 90 degrees.

2. Draw a small figure that has cyclic C_3 symmetry but not dihedral D_3 symmetry.



3. Give an example of a binary operation that is not associative. Give the operation and the underlying set.

   Subtraction on $\mathbb{Z}$

4. (Circle one: TRUE / FALSE). There exists a group G and elements $a, b, c \in G$ such that $b \neq c$ and $ab = ac$.

5. Give two *finite, nontrivial* subgroups of the nonzero complex numbers $\mathbb{C}^*$ under multiplication.

   $\{1, -1\}, \{1, i, -1, -i\}$

6. Determine the centralizer $C(F_1)$ of the reflection F_1 in the dihedral group D_3.

D_3	R_0	R_{120}	R_{240}	F_1	F_2	F_3
R_0	R_0	R_{120}	R_{240}	F_1	F_2	F_3
R_{120}	R_{120}	R_{240}	R_0	F_3	F_1	F_2
R_{240}	R_{240}	R_0	R_{120}	F_2	F_3	F_1
F_1	F_1	F_2	F_3	R_0	R_{120}	R_{240}
F_2	F_2	F_3	F_1	R_{240}	R_0	R_{120}
F_3	F_3	F_1	F_2	R_{120}	R_{240}	R_0

$C(F_1) = \{R_0, F_1\}$

7. How many cyclic generators are there for each of the following groups? (A *cyclic generator* of G is an element $g \in G$ such that $G = \langle g \rangle$.)

   (a) $\mathbb{Z}_5$ $\varphi(5)=4$

   (b) $\mathbb{Z}_6$ $\varphi(6)=2$

8. Suppose in 1-line permutation notation that $\alpha = [6\,2\,1\,5\,3\,4]$.

   (a) Write α in 2-line notation.

   (b) Write α in cycle notation.

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 1 & 5 & 3 & 4 \end{bmatrix}$

(b) $(1\ 6\ 4\ 5\ 3)(2)$

9. A permutation group $G \leq S_n$ on the elements $\{1, \ldots, n\}$ has order $|G| = 24$. The orbit $\{\sigma(1) \,|\, \sigma \in G\}$ of 1 has size 6. What is the size of the stabilizer $\{\sigma \in G \,|\, \sigma(1) = 1\}$ of 1?

   $24/6 = 4$

10. How many automorphisms are there of the group $\mathbb{Z}_5$?

    $|\mathrm{Aut}(\mathbb{Z}_5)| = |U(5)| = \varphi(5) = 4$

11. (Circle one: TRUE / FALSE) S_3 is isomorphic to D_3.

12. Let p be a prime. What do you know about a group G if $|G| = p$?

    G is cyclic

13. Let G be a group and let $H \leq G$. How many left cosets of H in G are also subgroups of G? (In other words, what is $|\{aH \mid a \in G,\, aH \leq G\}|$?)

1

14. How many elements of order 12 does $\mathbb{Z}_4 \times \mathbb{Z}_{15}$ have?

form: (a,b) $|a|=4$, $|b|=3$

$\varphi(4)=2$ $\varphi(3)=2$

$2 \cdot 2 = 4$

15. Write an external direct product of U groups that is isomorphic to $U(15)$.

$U(3) \oplus U(5)$

16. Write down a subgroup of D_4 that is *not* normal in D_4.

$\{R_0, V\} \ntriangleleft D_4$

17. (Circle one: TRUE / FALSE) For all positive integers n, $A_n \lhd S_n$.

18. Find H and K such that $\mathbb{Z}_{10} = H \times K$, $H \neq \mathbb{Z}_{10}$, and $K \neq \mathbb{Z}_{10}$.

$H = \langle 2 \rangle$, $K = \langle 5 \rangle$

19. (Circle one: TRUE / FALSE) If $\phi : G \to \overline{G}$ is a homomorphism, then ϕ must be onto.

20. Let $G = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mid n \in \mathbb{Z}_{\geq 0}, a_i \in \mathbb{R}\}$ be the group of finite degree polynomials with variable x over the reals, with group operation addition. Let $\phi : G \to G$ be the homomorphism defined by $\phi(f(x)) = f''(x)$, the second derivative of $f(x)$. What is the kernel of ϕ?

degree ≤ 1 polynomials with variable x over $\mathbb{R}$

$= \{a_1 x + a_0 \mid a_0, a_1 \in \mathbb{R}\}$

II. Constructions and Algorithms. (10 pts ea.) Do not write proofs, but do give clear, concise answers, including steps to algorithms where applicable.

21. Use the Euclidean Algorithm to find gcd(354, 126). Find $s, t \in \mathbb{Z}$ such that $\gcd(354, 126) = 354 \cdot s + 126 \cdot t$.

$354 = 2 \cdot 126 + 102$
$126 = 1 \cdot 102 + 24$
$102 = 4 \cdot 24 + \boxed{6}$
$24 = 4 \cdot 6 + 0$

$\gcd(354, 126) = 6$

Back solve for s, t:

$6 = 102 - 4 \cdot 24$
$24 = 126 - 1 \cdot 102$

$6 = 102 - 4(126 - 1 \cdot 102)$
$= 5 \cdot 102 - 4 \cdot 126$

$102 = 354 - 2 \cdot 126$

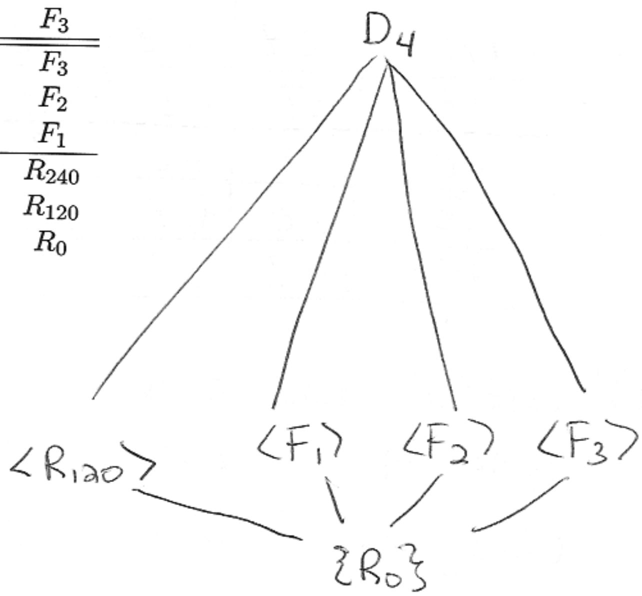
$6 = 5 \cdot (354 - 2 \cdot 126) - 4 \cdot 126$
$= 5 \cdot 354 - 14 \cdot 126$

$\boxed{s = 5 \quad t = -14}$

22. The subgroup diagram of a group G consists of points and line segments connecting certain points. For each distinct subgroup $H \leq G$, there is exactly one point. Two subgroups H and K are connected by a line segment provided that
(i) $H \leq K$, and
(ii) There is no third distinct subgroup J with $H \leq J \leq K$.

Construct the subgroup diagram for D_3 based on the given Cayley Table.

D_3	R_0	R_{120}	R_{240}	F_1	F_2	F_3
R_0	R_0	R_{120}	R_{240}	F_1	F_2	F_3
R_{120}	R_{120}	R_{240}	R_0	F_3	F_1	F_2
R_{240}	R_{240}	R_0	R_{120}	F_2	F_3	F_1
F_1	F_1	F_2	F_3	R_0	R_{120}	R_{240}
F_2	F_2	F_3	F_1	R_{240}	R_0	R_{120}
F_3	F_3	F_1	F_2	R_{120}	R_{240}	R_0



23. For this question, the group G has Cayley Table as given. Recall that $\text{Inn}(G) \approx G/Z(G)$.
(a) Compute the left cosets of $Z(G) = \{e, l\}$ in G.
(b) Use (a) to find $\text{Inn}(G)$, the group of inner automorphisms of G, without any additional computation. No repetitions – write down exactly $|\text{Inn}(G)|$ elements.
(c) Construct the Cayley Table for $G/Z(G)$.
(d) Explain from the Cayley Table in (c) whether $\text{Inn}(G)$ isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

G	e	i	j	k	l	m	n	o
e	e	i	j	k	l	m	n	o
i	i	l	k	n	m	e	o	j
j	j	o	l	i	n	k	e	m
k	k	j	m	l	o	n	i	e
l	l	m	n	o	e	i	j	k
m	m	e	o	j	i	l	k	n
n	n	k	e	m	j	o	l	i
o	o	n	i	e	k	j	m	l

(a) $\{e,l\}$, $i\{e,l\} = \{i,m\}$, $j\{e,l\} = \{j,n\}$, $k\{e,l\} = \{k,o\}$

(b) $\text{Inn}(G) = \{\varphi_e, \varphi_i, \varphi_j, \varphi_k\}$

(c)

$G/Z(G)$	eZ	iZ	jZ	kZ
eZ	eZ	iZ	jZ	kZ
iZ	iZ	eZ	kZ	jZ
jZ	jZ	kZ	eZ	iZ
kZ	kZ	jZ	iZ	eZ

(d) $G/Z(G)$ has order 4 but no elements of order 4. Therefore $G/Z(G) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
(Diagonal of table $\Rightarrow$ $x^2 = eZ$ for all $x \in G/Z(G)$.)

III. Proofs. (10 pts ea.) Part of the score is determined by careful formatting of the proof (forward and reverse directions, assumptions, conclusions, stating whether the proof is direct, contrapositive, contradiction, induction, etc.). Partial credit will be awarded for this as well.

24. Let G be a group and fix $x \in G$. Define the function $T_x : G \to G$ by $T_x(g) = xgx^2$. Either prove that T_x is always a bijection, or give a counterexample.

Proof T_x is one-to-one:

Let $h, g \in G$.

Assume $T_x(h) = T_x(g)$

$xhx^2 = xgx^2$ by defn.

$hx^2 = gx^2$ left canc. of x

$h = g$ right canc. of x^2

Therefore T_x is one-to-one.

T_x is onto:

Let $y \in G$.

Let $g = x^{-1}yx^{-2}$.

Then $T_x(g) = xgx^2$

$= xx^{-1}yx^{-2}x^2$

$= y$

and T_x is onto.

$\therefore$ T_x is a bijection. $\square$

25. Let $H \leq \mathbb{Z}$ and assume $|H| \geq 2$. Prove that $H \approx \mathbb{Z}$.

(Direct) $|H| \geq 2 \Rightarrow \{0, n\} \subseteq H$ where $n \neq 0$.
By closure under inverses, we may assume $n > 0$.
$|n| = \infty$, since all multiples of n are distinct.
Therefore $|H| = \infty$.
By the Fundamental Theorem of Cyclic groups, every subgroup of a cyclic group is cyclic. Therefore $H = \langle h \rangle$ for some $h \in H$, and $|h| = |H| = \infty$.
The unique cyclic group of infinite order up to isomorphism is $\mathbb{Z}$. $\square$

Prove ***ONE*** out of 26-27. Clearly indicate which proof you want graded.

26. Recall that D_6 is the dihedral group of plane symmetries of the regular hexagon.
(a) Write down $Z(D_6)$, $|D_6/Z(D_6)|$, and a list of all possible groups of order $|D_6/Z(D_6)|$ up to isomorphism. You do not have to justify this part.
(b) Determine what group in the list in part (b) that $D_6/Z(D_6)$ is isomorphic to. Justify each step in this part. (Hint: There are at least 2 approaches.)

27. Let $N \lhd G$ and $N \lhd H \leq G$. Prove that $H/N \lhd G/N$ if and only if $H \lhd G$.

26. (a) $Z(D_6) = \{R_0, R_{180}\}$ since 6 even.
$|D_6/Z(D_6)| = |D_6|/|Z(D_6)| = 12/2 = 6.$
$|G| = 6 \Rightarrow G \approx \mathbb{Z}_6$ or D_3.

(b) Assume to the contrary that $D_6/Z(D_6) \approx \mathbb{Z}_6$, which is cyclic.
(by contradiction)
By a Chapter 9 Theorem, $D_6/Z(D_6)$ cyclic $\Rightarrow D_6$ Abelian $\Rightarrow D_6/Z(D_6) \approx \mathbb{Z}_1$ ※
Therefore $D_6/Z(D_6) \approx D_3$. □

27. (Direct) ($\Rightarrow$) Assume $H/N \lhd G/N$. To show $H \lhd G$,
Let $h \in H$ and $g \in G$.
$gN\,hN(gN)^{-1} = ghg^{-1}N \in H/N$ by assumption.
Thus $ghg^{-1}N = h'N$ for some $h' \in H$.
By p138 part 4, $h'^{-1}ghg^{-1} \in N$, or $ghg^{-1} \in h'N$,
but $N \leq H$ and so by closure $ghg^{-1} \in H$.
($\Leftarrow$) Assume $H \lhd G$. To show $H/N \lhd G/N$,
let $hN \in H/N$ and $gN \in G/N$.
$ghg^{-1} \in H$ by assumption.
Thus $ghg^{-1} = h'$ for some $h' \in H$, and
$gN\,hN(g^{-1}N)^{-1} = ghg^{-1}N = h'N \in H/N$,
and so by Normal Subgroup test $H/N \lhd G/N$. □