

PRINT Last name:__________________ First name:__________________

Signature:__________________ Student ID:__________________

Math 430 Exam 3, Fall 2006

I. Examples, Counterexamples and short answer. (7 pts ea.) Do not give proofs, but clearly indicate your proposed example or counterexample, or short answer where appropriate.

1. Find a group G which can be written nontrivially as an internal direct product of 3 subgroups. (I.e., $G = H_1 \times H_2 \times H_3$ with $|H_1| > 1$, $|H_2| > 1$, and $|H_3| > 1$.)

$$\mathbb{Z}_{30} = \langle 2\rangle \times \langle 3\rangle \times \langle 5\rangle$$
$$\langle 15\rangle \times \langle 10\rangle \times \langle 6\rangle$$

2. Find a homomorphism $\phi : \mathbb{Z}_{18} \to \mathbb{Z}_{54}$ such that $\phi(2) = 18$. Determine Kerϕ and $\phi(\mathbb{Z}_{18})$ for this homomorphism.

$\varphi(2) = 18$
$\varphi(\cancel{4}) = 36$
$\varphi(\cancel{6}) = 0$

try $\text{Ker}\,\varphi = \langle 6\rangle$, $\varphi(\cancel{\mathbb{Z}_{18}}\,1)$ 9, so that

$\varphi(\mathbb{Z}_{18}) = \langle 9\rangle$.

3. Give an example of a cyclic group of the form $G \oplus H$, where $|G| > 1$ and $|H| > 1$.

$\mathbb{Z}_2 \oplus \mathbb{Z}_3$

4. Give an example of a *non-Abelian* group G and 4 distinct normal subgroups $N_1, \ldots, N_4 \lhd G$.

$G = D_4$

$N_1 = \{R_0\}$

$N_2 = \{R_0, R_{180}\}$

$N_3 = \langle R_{90} \rangle$

$N_4 = D_4$

5. **(i)** Give an example of a linear operator homomorphism $A : \mathbb{R}^3 \to \mathbb{R}^3$ (i.e., a 3×3 matrix A over the reals) such that

   (a) the kernel Ker$A$ is nontrivial (i.e., is neither just the 0-vector nor all of $\mathbb{R}^3$), and

   (b) the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not in the image of A.

$$A(\mathbb{R}^3) = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\} \approx \mathbb{R}^2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6. Give an example of a group G, a subgroup $H \leq G$, and a coset aH such that aH can be interpreted as a line passing through the point $(3, 5)$ in Cartesian coordinates.

$G = \mathbb{R}^2$ under addition

$H = \{ (x, x) : x \in \mathbb{R} \}$

$a = (3, 5)$

II. Constructions and Algorithms. (14 pts ea.) Do not write proofs, but do give clear, concise answers, including steps to algorithms where applicable.

7. Determine the number of subgroups of order 12 in the group $\mathbb{Z}_{20} \oplus \mathbb{Z}_{18} \oplus \mathbb{Z}_{75}$. Express each subgroup in the form $\langle \ldots \rangle \oplus \langle \ldots \rangle \oplus \langle \ldots \rangle$.

$\langle 5 \rangle \oplus \langle 6 \rangle \oplus \langle 0 \rangle$

$\langle 10 \rangle \oplus \langle 3 \rangle \oplus \langle 0 \rangle$ 4

$\langle 5 \rangle \oplus \langle 0 \rangle \oplus \langle 25 \rangle$

$\langle 10 \rangle \oplus \langle 9 \rangle \oplus \langle 25 \rangle$

8. Regroup the rows/columns of the Cayley table below with respect to the cosets of the subgroup $\{e, x\}$. Determine whether $\{e, x\}$ is a normal subgroup, giving an explicit reason from the Cayley table you construct.

	e	v	w	x	y	z
e	e	v	w	x	y	z
v	v	w	e	y	z	x
w	w	e	v	z	x	y
x	x	z	y	e	w	v
y	y	x	z	v	e	w
z	z	y	x	w	v	e

	e	x	v	z	w	y
e	e	x	v	z	w	y
x	x	e	z	v	y	w
v	v	y	w	x	e	z
z	z	w	y	e	x	v
w	w	z	e	y	v	x
y	y	v	x	w	z	e

If $\{e,x\}$ is normal, then mult in the factor group is well-defined.

but $vz = x \in x\{ex\} = \{e,x\}$

$zv = y \in y\{ex\} = \{y,v\}$ so $\{e,x\}$ is not normal.

III. Proofs. (15 pts ea.) Part of the score is determined by careful formatting of the proof (forward and reverse directions, assumptions, conclusions, stating whether you are proving by direct proof, contrapositive, contradiction, induction, etc.). Partial credit will be awarded for this as well.

Prove ***ONE*** of 9-10. Clearly indicate which proofs you want graded.

9. Suppose that G is a finite Abelian group that has at least 3 elements of order 3. Prove that 9 divides $|G|$. (You may use, for example, $N, M \lhd G \Rightarrow NM \lhd G$, but this is not the only method.)

10. Prove or disprove the following. $\text{stab}_{S_n}(1) \lhd S_n$ (that is, the stabilizer of 1 is a normal subgroup of S_n).

9. Let $a, b \in G$ such that $|a| = 3 = |b|$ and $b \notin \{a, a^{-1}\}$.

Then $\langle a \rangle, \langle b \rangle \lhd G$ and $\langle a \rangle \cap \langle b \rangle = \{e\}$.

$\therefore$ $\langle a \rangle \times \langle b \rangle \lhd G$ with order 9.

By Lagrange's Thm, $9 \mid |G|$.

10. $\text{Stab}_{S_n}(1) = \{(1), (23), (24), (34), (234), (243)\}$

$(12)(23)(12) = (13)(2) \notin \text{Stab}_{S_n}(1)$

$\Rightarrow$ not normal.

Prove ***ONE*** out of 11-12. Clearly indicate which proof you want graded.

11. Classify the homomorphic images of D_4; that is, find all possible groups $\phi(D_4)$ up to isomorphism when $\phi : D_4 \to \overline{G}$ is a homomorphism.

12. Let $\mathbb{C}^*$ be the nonzero complex numbers under multiplication with subgroup $U = \{x \in \mathbb{C} : |x| = 1\}$. Let $\mathbb{R}^+$ be the positive reals under multiplication. Recall that for any two complex numbers $a + bi, c + di \in \mathbb{C}$, $|(a + bi)(c + di)| = |a + bi||c + di|$. Now prove that $\mathbb{C}^*/\mathbb{R}^+ \approx U$.

11. normal subgroups are $\langle R_0 \rangle$, $\langle R_{90} \rangle$, $\langle R_{180} \rangle$, $\langle R_{180}, F \rangle$ where F is a reflection and D_4.

The possible images up to isom are D_4/N where $N \triangleleft D_4$.

$D_4/\langle R_0 \rangle \approx \boxed{D_4}$ $\quad D_4/\langle R_{90} \rangle \approx \boxed{\mathbb{Z}_2}$ $\quad D_4/\langle R_{180} \rangle \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

$D_4/\langle R_{180}, F \rangle \approx \mathbb{Z}_2$, $\quad D_4/D_4 \approx \boxed{\{e\}}$.

$H\langle R_{180} \rangle = V\langle R_{180} \rangle$ $\qquad H\langle R_{180} \rangle D\langle R_{180} \rangle = R_{90}\langle R_{180} \rangle$

$D\langle R_{180} \rangle = D'\langle R_{180} \rangle$

$R_{90}\langle R_{180} \rangle = R_{270}\langle R_{180} \rangle$

12. Let $\varphi : \mathbb{C}^* \to U$ be defined by $\varphi(x) = \frac{x}{|x|}$.

$\ker \varphi = \{x : \frac{x}{|x|} = 1\} = \{\frac{a+bi}{\sqrt{a^2+b^2}} : \frac{a+bi}{\sqrt{a^2+b^2}} = 1\}$

require $b = 0$. so $\qquad \therefore \mathbb{C}^*/\ker \varphi \approx U$

$= \{\frac{a}{\sqrt{a^2}} : \frac{a}{\sqrt{a^2}} = 1\} = \mathbb{R}^+$. $\qquad \mathbb{C}^*/\mathbb{R}^+ \approx U$.

onto: Let $a+bi \in U$ ~~[illegible]~~. Then $\varphi(a+bi) = \frac{a+bi}{|a+bi|} = a+bi$. onto.