Motivation

Just as definite integrals can be solved using antiderivatives by the Fundamental Theorem of Calculus, we also want a method of solving finite summations of the form \( \sum_{i=1}^{n} f(i) \). The summations appear frequently in mathematics: in finite difference methods used to solve differential equations numerically on a computer, in combinatorial identities that count discrete objects by grouping them according to type, and in countless theorems for which a closed form of a summation is useful.

Finite Difference Formula

In calculus we have the familiar derivative formula
\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]
For discrete applications, we may not be able to let \( h \) go to 0. For example, \( f \) might be defined only for integer \( x \), or perhaps we are implementing a numerical procedure which samples \( f \) only at values of \( x \) with spacing like the integers. In this case we may alternatively set \( h = 1 \) and define the finite (forward) difference
\[
\Delta f(x) = f(x + 1) - f(x).
\]
If we take, for integer \( m \), the definition of the falling power of \( x \) to be
\[
x^m = \begin{cases} 
  x(x - 1) \cdots (x - m + 1) & \text{if } m > 0, \\
  1 & \text{if } m = 0, \\
  \frac{1}{(x+1) \cdots (x+(-m))} & \text{if } m < 0;
\end{cases}
\]
then it is a straightforward exercise in algebra to derive the following finite difference analogy to the power rule \( (d/dx)x^m = mx^{m-1} \):
\[
\Delta x^m = \begin{cases} 
  mx^{m-1} & \text{if } m \neq 0, \\
  0 & \text{if } m = 0.
\end{cases} \tag{1}
\]

A Word on Summation Notation

We will be using two forms of summation notation in what follows. First, is the notation we are accustomed to, namely
\[
\sum_{k=1}^{n} g(k) = g(1) + g(2) + \cdots + g(n-1) + g(n).
\]
Here the index \( k \) ranges from 1 up to \( n \), hitting every integer value in between and both endpoints 1 and \( n \).

The second notation is a formal notation for defining the analogy to a definite integral, namely the definite sum
\[
\sum_{a}^{b} g(x) \delta x.
\]
We don’t know this now before reading the rest of these notes, but the relationship between these two sums is
\[
\sum_{1}^{n} g(x) \delta x = \sum_{x=1}^{n-1} g(x) = g(1) + g(2) + \cdots + g(n - 1).
\]
In other words, the top index \( n \) is missing when we transfer from the second formal notation to the usual summation notation we already know about.

Antiderivatives and Antidifferences

In calculus we define antiderivatives by
\[
g(x) = f'(x) \quad \text{if and only if} \quad \int g(x) \, dx = f(x) + C,
\]

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where $C$ is any constant. In other words, $f(x) + C$ is the class of functions whose derivatives are all $g(x)$. By analogy, we formally define the indefinite sum $\sum g(x)\delta x$ by

$$g(x) = \Delta f(x) \quad \text{if and only if} \quad \sum g(x)\delta x = f(x) + C;$$

where $C$ could be a constant, or since the finite difference is defined with $h = 1$ spacing, $C$ could be any periodic function with $C(x) = C(x + 1)$ for all $x \in \mathbb{R}$, such as $\sin 2\pi x$. The formal sum $\sum g(x)\delta x$ is still vague, but we can make it more concrete by making an analogy with definite integrals.

**Definite Integrals and Definite Sums**

In calculus we define definite integrals by

$$\int_a^b g(x) \, dx = f(x)|_a^b = f(b) - f(a),$$

whenever $f(x)$ is an antiderivative of $g(x)$. Now suppose $f(x)$ is an antidifference of $g(x)$, and let us consider what we might get for finite calculus. First we need notation. Let us decide by fiat that the notation we want is $\sum_a^b g(x)\delta x$, and we want to define this sum so that, analogous to definite integrals,

$$\sum_a^b g(x)\delta x = f(b) - f(a).$$

When $b = a$, we would like to get $\sum_a^a g(x)\delta x = f(a) - f(a) = 0$, analogously to $\int_a^a g(x) \, dx = 0$. When $b = a + 1$, we would like to get $\sum_a^{a+1} g(x)\delta x = f(a + 1) - f(a)$, which by the definition of $g(x) = \Delta f(x)$, gives

$$\sum_a^{a+1} g(x)\delta x = f(a + 1) - f(a) = g(a).$$

Now say that $b = a + k$, where $m$ is some positive integer. Then we have

$$\sum_a^{a+m} g(x)\delta x = f(a + m) - f(a)$$

$$= [f(a + m) - f(a + m - 1)] + [f(a + m - 1) - f(a + m - 2)] + \cdots + [f(a + 2) - f(a + 1)] + [f(a + 1) - f(a)]$$

$$= g(a + m - 1) + g(a + m - 2) + \cdots + g(a + 1) + g(a).$$

The middle line is just a telescoping sum that we insert to make things work out. We see that this is exactly what we need to define the definite sum as a concrete object.

$$\sum_a^{b} g(x)\delta x : = \sum_{x=a}^{b-1} g(x) = g(a) + g(a + 1) + \cdots + g(b - 1). \quad (2)$$

Notice the sum goes to $b - 1$ instead of to $b$; this is required to make the analogy with definite integrals work.

**Table of Finite Differences and Indefinite Sums**

Define the harmonic number function

$$H(x) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x},$$

when $x$ is an integer. Furthermore, define the “right-shift” operator

$$Ef(x) = f(x + 1).$$

We have the following table of differences and sums, like an integral table.
Table of finite differences and indefinite sums

<table>
<thead>
<tr>
<th>$f$</th>
<th>$\Delta f = g$</th>
<th>$f$</th>
<th>$\Delta f = g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 = 1$</td>
<td>$0$</td>
<td>$2x$</td>
<td>$2x$</td>
</tr>
<tr>
<td>$x^1 = x$</td>
<td>$1$</td>
<td>$c^x$</td>
<td>$(c - 1)c^x$</td>
</tr>
<tr>
<td>$x^m$</td>
<td>$mx^{m-1}$</td>
<td>$c \cdot f$</td>
<td>$c \Delta f$</td>
</tr>
<tr>
<td>$x^{m+1} / (m+1)$</td>
<td>$x^m$</td>
<td>$f + g$</td>
<td>$\Delta f + \Delta g$</td>
</tr>
<tr>
<td>$H(x)$</td>
<td>$x^{-1}$</td>
<td>$fg$</td>
<td>$f \Delta g + Eg \Delta f$</td>
</tr>
</tbody>
</table>

Recognizable in the right-hand column are the constant, sum, and product rules of finite difference, analogous to differentiation. Also, recall that reversing the product rule gives us a sum by parts (cf. integration by parts) formula:

$$\sum u \Delta v = uv - \sum E v \Delta u$$

which can be made into a definite sum formula:

$$\sum_{a}^{b} u \Delta v = u(b - 1)v(b - 1) - u(a)v(a) - \sum_{a}^{b} E v \Delta u$$

Binomial Coefficients

Finally, recall that the binomial coefficients are defined for integer $n$ and $k$ as

$$\binom{n}{k} = \begin{cases} \frac{n!}{(n - k)!k!} & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } n < k. \end{cases}$$

We have the following simple binomial identity:

$$\binom{n + 1}{k} = \binom{n}{k} + \binom{n}{k - 1}$$

Project Exercises

1. Prove the finite difference power rule in (1).
2. Prove the sum rule $\Delta(f + g) = \Delta f + \Delta g$ of the table.
3. Verify the identity $x^2 = x^2 + x^1$, and use it to compute the sum $\sum_{k=0}^{n-1} k$ using finite calculus facts.
4. Prove $\Delta H(x) = x^{-1}$ (you may assume $x$ is a positive integer).
5. Prove the two exponential rules in the table, i.e., $\Delta 2^x = 2^x$ and $\Delta c^x = (c - 1)c^x$. Explain two connections (analogies or differences) with the usual calculus or formulas you are familiar with.
6. Use the definite sum by parts formula in (4) to evaluate $\sum_{k=1}^{n-1} k2^k$.
7. Compute $\Delta \left(\binom{n}{k}\right)$, and use (5) to compute $\sum_{x=x_0}^{x_{n-1}} (x)$.
8. Suppose that instead of defining the finite difference as $\Delta f(x) = f(x + 1) - f(x)$, that we define $\nabla f(x) = f(x) - f(x - 1)$. Can we define finite difference and indefinite sum formulas analogously to the table?
9. Suppose that instead of defining the finite difference as $\Delta f(x) = f(x + 1) - f(x)$, we define it as the “middle difference” $\Delta f(x) = f(x + 1/2) - f(x - 1/2)$. What is the second finite difference $\Delta^2 f(x)$ of $f(x)$ (analogous to the second derivative)? Can we find any nice formulas for second derivatives under this assumption?

References