Practice Exam 2, Revised 4/17/2009 for Sections 2.3-4.2

On Exam 2, show work for full credit. Partial credit for good proof structure even if proof is not correct.

1. Let $f(n) = 2n + 1$. Is f a one-to-one function from the set of integers to the set of integers? Is f an onto function from the set of integers to the set of integers? Is f increasing, decreasing, strictly increasing, or strictly decreasing (there may be more than one answer)? Give proofs.
2. Suppose that f is the function from the set $\{a, b, c, d\}$ to itself with $f(a) = d$, $f(b) = a$, $f(c) = b$, $f(d) = c$. Find the inverse of f.
3. Find a formula that generates the following sequence $a_1, a_2, a_3, \ldots$.
4. $5, 9, 13, 17, 21, \ldots$.
5. $1, 1/3, 1/5, 1/7, 1/9, \ldots$.
6. $0, 2, 0, 2, 0, 2, 0, \ldots$.
7. Find the sum $1 - 1/2 + 1/4 - 1/8 + 1/16 - \ldots$.
8. Find the sum $112 + 113 + 114 + ... + 673$.
9. Find $\sum_{j=1}^{3} \sum_{i=1}^{j} ij$.
10. Rewrite $\sum_{i=-3}^{4}(i^2 + 1)$ so that the index of summation has lower limit 0 and upper limit 7.
11. Write pseudocode for an algorithm that takes a list of n integers $a_1, a_2, \ldots, a_n$ and finds the number of integers each greater than five in the list.
12. Describe how the binary search algorithm searches for 27 in the following list: 5 6 8 12 15 21 25 31.
13. Prove that $1^2 + 2^2 + \ldots + n^2$ is $O(n^3)$. (Technically speaking this requires induction, but you can skip any inductive proof.)
14. Find witnesses C and k from the definition of big-oh to show that $f(n) = 3n^2 + 8n + 7$ is $O(n^2)$. (Hint: $x > k \to |f(n)| \leq C|n^2|$.)

In the questions below write the best big-oh notation to describe the complexity of the algorithm.

15. A binary search of n elements.
16. A linear search to find the smallest number in a list of n numbers.
17. An algorithm that lists all ways to put the numbers $1, 2, 3, \ldots, n$ in order.
18. An algorithm that prints all bit strings of length n.

In the questions below find the best big-oh notation to describe the number designated steps of the algorithm.

19. The number of print statements in the following:

```
i := 1
j := 1
while i ≤ n
begin
    while j ≤ i
    begin
        print "hello"
        j := j + 1
    end
    i := i + 1
end
```

20. The number of comparisons in the best case, average case, and worst case analysis of linear search (p.170).

21. Prove or disprove: For all integers a, b, c, d, if $a|b$ and $c|d$, then $(a+c)|(b+d)$.

22. Prove or disprove: For all integers a, b, c, if $a|b$ and $b|c$ then $a|c$.

23. Prove or disprove: For all integers a, b, c, if $a|bc$, then $a|b$ or $a|c$.

24. Prove or disprove: For all integers a, b, c, if $a|c$ and $b|c$, then $ab|c^2$.

Use the improved algorithm discussed in class to find the prime factorizations of the following.

25. Find the prime factorization of 510,510.

26. List all positive integers less than 30 that are relatively prime to 20.

27. Find $\gcd(20!, 12!)$ and $\text{lcm}(20!, 12!)$.

28. Find $\gcd(2^{89}, 2^{346})$ and $\text{lcm}(2^{89}, 2^{346})$.

29. Suppose that the $\text{lcm}(a, b) = 400$ and $\gcd(a, b) = 10$. If $a = 50$, find b.

30. Applying the division algorithm with $a = -41$ and $d = 6$ yields what value of r?

31. Find 18 mod 7.

32. Find the hexadecimal expansion of $ABC_{16} + 2F5_{16}$.

33. Prove or disprove: If p and q are primes, both > 2, then $p + q$ is composite.

34. Find the smallest positive integer a such that $a+1 \equiv 2a \pmod{11}$.

35. Prove or disprove. Let a, b, c, d, and m be integers with $m > 1$. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv b+d \pmod{m}$.

36. Prove or disprove. Let a, b, c, d, and m be integers with $m > 1$. If $a \equiv b \pmod{m}$, then $2a \equiv 2b \pmod{m}$.

37. A message has been encrypted using the function $f(x) = (x+5) \bmod 26$. If the message in coded form is $JCFHY$, decode the message.

38. Use the Euclidean algorithm to find $\gcd(900, 140)$.

39. Suppose you wish to use the Principle of Mathematical Induction to prove that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1$ for all $n \geq 1$.
(a) Write $P(1)$
(b) Write $P(5)$
(c) Write $P(k)$
(d) Write $P(k+1)$
(e) Use the Principle of Mathematical Induction to prove that $P(n)$ is true for all $n \geq 1$.

40. Use the Principle of Mathematical Induction to prove that $1 + 3 + 9 + 27 + \cdots + 3^n = \frac{3^{n+1}-1}{2}$ for all $n \geq 0$.

41. Use the Principle of Mathematical Induction to prove that $2n + 3 \leq 2^n$ for all $n \geq 4$.

42. Use the Principle of Mathematical Induction to prove that $3|(n^3 + 3n^2 + 2n)$ for all $n \geq 1$.

43. Use the Principle of Mathematical Induction to prove that any integer amount of postage from 18 cents on up can be made from an infinite supply of 4-cent and 7-cent stamps.

44. Use mathematical induction to show that n lines in the plane passing through the same point divide the plane into 2n regions.

45. Find the error in the following proof of this "theorem":
"Theorem: Every positive integer equals the next largest positive integer."
"Proof: Let $P(n)$ be the proposition '$n = n+1$'. To show that $P(k) \to P(k+1)$, assume that $P(k)$ is true for some k, so that $k = k+1$. Add 1 to both sides of this equation to obtain $k+1 = k+2$, which is $P(k+1)$. Therefore $P(k) \to P(k+1)$ is true. Hence $P(n)$ is true for all positive integers n."

In the questions below give a recursive definition with initial condition(s).

46. The function $f(n) = 2^n$, $n = 1, 2, 3, \ldots$.

47. The function $f(n) = 5n + 2$, $n = 1, 2, 3, \ldots$.

48. Find $f(2)$ and $f(3)$ if $f(n) = 2f(n-1) + 6$, $f(0) = 3$.