I. Short answer (1.5 pts each). No partial credit – only the response will be graded. Suggested time 1 hour.

1. Find functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g$ has the rule $(f \circ g)(x) = \lfloor x^2 + 7 \rfloor$.
   - **(a)** $g(x) = x^2$, $f(x) = \lfloor x \rfloor + 7$.
   - **(b)** $g(x) = \lfloor x \rfloor + 7$, $f(x) = x^2$.
   - **(c)** $g(x) = \lfloor x \rfloor$, $f(x) = x^2 + 7$.
   - **(d)** $g(x) = x^2 + 7$, $f(x) = \lfloor x \rfloor$.
   - **(e)** $g(x) = x + 7$, $f(x) = \lfloor x^2 \rfloor$.
   - **(f)** $g(x) = \lfloor x + 7 \rfloor$, $f(x) = x^2$.

2. Circle the letter for the correct statement about the function $f : \mathbb{Z} \to \mathbb{Z}$ with rule $f(x) = 2x$.
   - **(a)** f is one-to-one and onto.
   - **(b)** f is onto but not one-to-one.
   - **(c)** f is one-to-one but not onto.
   - **(d)** f is neither one-to-one nor onto.

3. Suppose $g : \mathbb{R} \to \mathbb{R}$ has the following property for all real numbers x and y: if $x < y$ then $g(x) > g(y)$. (I.e., g is strictly decreasing.) Which of the following is true?
   - **(a)** g must be 1-1 but is not necessarily onto $\mathbb{R}$.
   - **(b)** g is onto $\mathbb{R}$ but is not necessarily 1-1.
   - **(c)** g must be both 1-1 and onto $\mathbb{R}$.
   - **(d)** g is not necessarily 1-1 and not necessarily onto $\mathbb{R}$.

4. Write down the initial term and common ratio of the series $\sum_{i=3}^{\infty} \frac{2 \cdot 4^i}{5^i}$.

   initial term: $2 \cdot \frac{4^3}{5^3} = \frac{128}{75}$ common ratio: $\frac{4}{5}$

   $\frac{2 \cdot 4^3}{5^3} + \frac{2 \cdot 4^4}{5^4} + \cdots$

5. Write down the initial term and common difference of the progression $(4, -3, -10, -17, \ldots)$.

   initial term: 4 common difference: -7

6. Suppose $f : \mathbb{Z} \to \mathbb{R}$ has the rule $f(n) = 2n - 3$. Circle the letter corresponding to the range of f.
   - **(a)** the set of natural numbers $\{0, 1, 2, \ldots\}$
   - **(b)** $\mathbb{Z}$
   - **(c)** the set of odd integers
   - **(d)** the set of even integers
   - **(e)** the real numbers
   - **(f)** the rational numbers

Total 9

7. Using the Binomial Theorem, find the sum of the series $\sum_{j=0}^{n} \binom{n}{j}(-1)^{n-j}$.

(a) 2^n (b) $2^n/2$

(c) 21 (d) -1

(e) 0 (f) 1

$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$

$x=-1, \quad y=1$

8. What is the probability of rolling a multiple of 3 on a fair 6-sided die?

(a) 1/6 (b) 2/6 (c) 3/6

(d) 4/6 (e) 5/6 (f) 1

9. Recall the following properties of a relation $R \subseteq A \times A$ (saying $a\,R\,b$ is the same as saying $(a,b) \in R$:

2pts

Reflexive:	For all $a \in A$, $a\,R\,a$.
Symmetric:	For all $a, b \in A$, if $a\,R\,b$, then $b\,R\,a$.
Antisymmetric:	For all $a, b \in A$, if $a\,R\,b$ and $b\,R\,a$, then $a = b$.
Transitive:	For all $a, b, c \in A$, if $a\,R\,b$ and $b\,R\,c$, then $a\,R\,c$.

Check the boxes to the left of the properties satisfied by the relation $R_< = \{(x,y) \in \mathbb{R}^2 \mid x < y\}$.

- [ ] reflexive
- [ ] symmetric
- [x] antisymmetric
- [x] transitive

10. A fast food restaurant offers 3 types of items: tacos, hamburgers, and chicken. Assume there is an unlimited supply of each type of item. In how many ways can a very hungry person buy 5 items if the order of the items does not matter?

(a) $P(5,3)$ (b) $C(5,3)$

(c) $P(7,2)$ (d) $C(7,2)$

(e) 5^3

tacos | hamburgers | chicken

o o o
o o

5 balls in 3 bins

$C(5+3-1, 3-1)$

11. What is the minimum number of persons in a group necessary to guarantee that at least 4 of them were born on the same day of the week?

(a) 13 (b) 22

(c) 20 (d) 4

(e) 28 (f) 21

$N = (k-1) \cdot n + 1$

$N = 3 \cdot 7 + 1 = 22$

Total 8

12. Recall that the power set $\mathcal{P}(S)$ of the set S is defined to be the set containing all subsets of S. Let $S = \{1, 2, 3, 4\}$. In the blank to the left of each statement, circle **T** if the statement is true, and **F** if false. (Therefore this question has 5 answers!!)

T **F** (iii) $\{1,3\} \in \mathcal{P}(S)$ **T** **F** (ii) $\{1,3\} \subseteq \mathcal{P}(S)$ **T** **F** (i) $\{\{2\}\} \subseteq \mathcal{P}(S)$

T **F** (v) $\{4\} \subseteq \mathcal{P}(S)$ **T** **F** (iv) $\{\{2\},\{4\}\} \subseteq \mathcal{P}(S)$

13. Which of these assertions is correct concerning the statement "If x^3 is irrational, then x is irrational?"

(a) This statement is true, as can be shown most easily using a direct proof.

(b) This statement is true, as can be shown most easily using a proof by contraposition.

(c) This statement is false because a counterexample can be found.

(d) This statement is false because the negation of the statement can be proved easily by contradiction.

14. Suppose you want to give a proof by contrapositive of this result for all integers:
"If a is odd and b is even, then $a + b$ is odd."
Circle the letter of the assumption you would begin the proof with.

(a) a is odd and b is even. (b) $a + b$ is even. (c) a is even or b is odd.

(d) a is even and b is odd. (e) $a + b$ is odd.

15. Suppose you want to prove a theorem about the product of absolute values of real numbers $|x| \cdot |y|$. If you were to give a proof by cases, what set of cases would probably be the best to use?

(a) $x > y$; $x < y$; $x = y$

(b) x divides y; y divides x; $\gcd(x, y) = 1$

(c) Both x and y nonnegative; one negative and one nonnegative; both negative

(d) Both x and y rational; one rational and one irrational; both irrational.

(e) Both x and y even; one even and one odd; both odd.

16. According to De Morgan's laws, $\overline{R \cup (S \cap T)} =$

$\overline{R} \cap \overline{(S \cap T)} = \overline{R} \cap (\overline{S} \cup \overline{T})$

(a) $\overline{R} \cap (S \cap T)$. (b) $\overline{R} \cup (\overline{S} \cap \overline{T})$. (c) $\overline{R} \cap (\overline{S} \cup \overline{T})$.

(d) $\overline{R} \cup (S \cap T)$. (e) $\overline{R} \cap (\overline{S} \cap \overline{T})$.

Total 7.5

17. An algorithm prints out each bit in a bitstring of length n. What is a reasonable operation with which to measure the complexity of the algorithm, and what is the best big-oh notation for the number of these operations assuming the algorithm is efficient?

operation: print bit big-oh complexity: $O(n)$

18. Circle the letter corresponding to the true statement.

(a) If $f(x)$ is $\Omega(g(x))$ then $g(x)$ is $\Omega(f(x))$. (b) If $f(x)$ is $\Theta(g(x))$ then $g(x)$ is $\Theta(f(x))$.

(c) If $f(x)$ is $O(g(x))$ then $g(x)$ is $O(f(x))$. (d) If $f(x)$ is $\Omega(g(x))$ then $g(x)$ is $\Theta(f(x))$.

19. Is 133 prime? (Circle one.)

YES **NO**

$$3\overline{)133}^{\,4} \quad 12 \quad 13 \qquad 7\overline{)133}^{\,19} \quad 7 \quad 63$$

20. Compute $\gcd(2^3 \cdot 3^4 \cdot 7^3, 2^5 \cdot 5^2 \cdot 7^2)$.

Answer: $2^3 7^2$

21. Compute the following: $32 \mod 7 =$ 4 $\qquad -30 \mod 9 =$ 6

$32 = 4 \cdot 7 + 4 \qquad -30 = -4 \cdot 9 + 6$

22. Suppose that $P(n)$ is the statement "$n + 1 = n + 2$." What is wrong with the following "proof" that the statement $P(n)$ is true for all nonnegative integers n:

You assume that $P(k)$ is true for some positive integer k; that is, $k + 1 = k + 2$. Then you add 1 to both sides of this equation to obtain $k + 2 = k + 3$; therefore $P(k + 1)$ is true. By the principle of mathematical induction $P(n)$ is true for all nonnegative integers n.

(a) The proof is incorrect because the statement used in the inductive hypothesis is incorrect.

(b) There is nothing wrong with this proof.

(c) The proof is incorrect because you cannot add 1 to both sides of the equation in the inductive step.

(d) The proof is incorrect because there is no basis step.

23. For which of the following is the recursively defined set S equal to the set of odd positive integers?

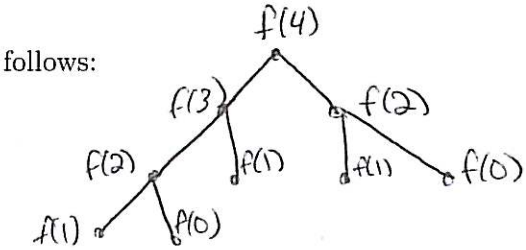
(a) $1 \in S$; $3 \in S$; $x \in S \rightarrow x + 4 \in S$. (b) $99 \in S$; $x \in S \rightarrow x - 2 \in S$

(c) $2 \in S$; $x \in S \rightarrow x + 2 \in S$ (d) $1 \in S$; $x \in S \rightarrow 2x + 1 \in S$

(e) None of these

Total 10.5

24. A recursive algorithm for computing the Fibonacci numbers is as follows:



procedure *fibonacci*(n : nonnegative integer)
if $n = 0$ **then** return 0
else if $n = 1$ **then** return 1
else return $fibonacci(n-1) + fibonacci(n-2)$

How many times is $fibonacci(1)$ called in order to compute $fibonacci(4)$?

(a) 0 (b) 1
(c) 2 (d) 3
(e) 5 (f) 8

25. How many different passwords are available if a password consists of 5 lowercase letters followed by 2 decimal digits (from $\{0, 1, \ldots, 9\}$)?

(a) $5 \cdot 26 + 2 \cdot 10$ (b) $26^5 + 10^2$ (c) $P(26,5) \cdot P(10,2)$
(d) $P(26,5) + P(10,2)$ (e) $C(26,5) \cdot C(10,2)$ (f) $26^5 \cdot 10^2$

$\underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{10} \cdot \underline{10}$

26. A standard deck of playing cards consists of 52 cards, which correspond to the set $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\} \times \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$, so that each card has one of 13 ranks and one of 4 suits. How many 5-card hands are a full house, that is, consisting of 3 cards of one rank and 2 cards of a second rank?

(a) $13 \cdot 12 \cdot C(4,3) \cdot C(4,2)$ (b) $C(13,2) \cdot C(4,3) \cdot C(4,2)$
(c) $P(13,2) \cdot P(4,3) \cdot P(4,2)$ (d) $13 \cdot 12 \cdot P(4,3) \cdot P(4,2)$

pick triple rank: 13 ways
pick pair rank: 12 ways
pick triple suits $C(4,3)$ ways
pick pair suits $C(4,2)$ ways
sum rule of counting

27. How many permutations of the letters in the word MISSISSIPPI are there?

(a) $P(11,4) \cdot P(11,4) \cdot P(11,2)$ (b) $11!/10$ (c) $11!/(2 \cdot 4! + 2!)$
(d) $C(11,4) \cdot C(11,4) \cdot C(11,2)$ (e) $11!/(4! \cdot 4! \cdot 2!)$

$$\frac{11!}{4!4!2!}$$

11 letters
M: 1 S: 4
I: 4 P: 2

28. What is the best big-oh notation for the number of comparisons used by mergesort to sort a list of n numbers?

Answer: $O(n \log n)$

Total 7.5

29. Suppose $Q(u, v, w)$ is a predicate where the universe for u, v, and w is $\{0, 1\}$. Also suppose that the predicate is true in the following cases – $Q(0, 0, 0)$, $Q(0, 0, 1)$, $Q(1, 0, 0)$, $Q(1, 1, 1)$ – and false otherwise. Circle the letter of the true quantified statement.

(a) $\exists u \exists v \forall w\ Q(u, v, w)$

(b) $\exists u \exists w \forall v\ Q(u, v, w)$

(c) $\exists u \forall v \forall w\ Q(u, v, w)$

(d) $\forall u \exists w \forall v\ Q(u, v, w)$

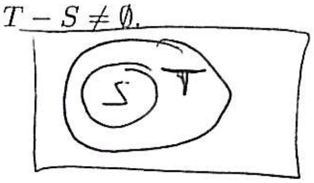
30. Circle the letter of the negation of $\forall x \exists y\ (x < y)$.

(a) $\exists x \exists y\ (x \geq y)$ (b) $\exists x \forall y\ (x \geq y)$ (c) $\forall x \exists y\ (x \geq y)$ (d) $\exists x \forall y\ (x < y)$

31. Circle the letter of the statement which makes the implication about sets true. If $S \subseteq T$, then

(a) $\overline{T} \subseteq \overline{S}$. (b) $T \subseteq S \cap T$. (c) $T - S \neq \emptyset$.

(d) $T - S \subseteq S - T$. (e) $S \cup T = S \cap T$.



32. Circle the letter of the inverse of the statement "If it is a warm day, then I go hiking."

(a) If I go hiking, then it is a warm day.

(b) If it is not a warm day, then I do not go hiking.

(c) If I do not go hiking, then it is not a warm day.

(d) It is a warm day, and I do not go hiking.

33. Assume that a, b, and c are all positive integers larger than 1. Circle the letter of the negation of the statement "a is prime, and b and c are composite."

(a) a is composite and b and c are prime.

(b) a is composite, and either b is prime or c is prime.

(c) If a is composite, then b is prime or c is prime.

(d) a is composite or b is prime or c is prime.

(e) Either a is composite, or b is prime and c is prime.

$p \wedge q \wedge r$

p: a prime

q: b composite

r: c composite

Total 7.5

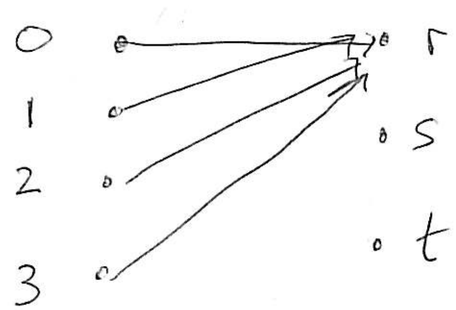
Part II. Computation, Algorithms, and Examples (5 pts ea.). Show work for full credit. Suggested time 25 minutes.

34. Use the Euclidean algorithm to find $\gcd(132, 102)$.

$$132 = 1 \cdot 102 + 30$$
$$102 = 3 \cdot 30 + 12$$
$$30 = 2 \cdot 12 + \boxed{6}$$
$$12 = 2 \cdot 6 + 0$$

$$\gcd = 6$$

35. Give an example of a function with domain $\{0, 1, 2, 3\}$ and codomain $\{r, s, t\}$ which is not one-to-one and not onto.



36. Trace through the Insertion Sort algorithm on the list 2, 4, 1, 3 by writing down the order of the list after each increment of j or i.

procedure insertionSort($a_1, \ldots, a_n$: reals with $n \geq 2$)
for $j := 2$ **to** n
begin
  $i := 1$
  while $a_j > a_i$
    $i := i + 1$
  $m := a_j$
  for $k := 0$ **to** $j - i - 1$
    $a_{j-k} := a_{j-k-1}$
  $a_i := m$
end

Trace

initial list:	2	1	4	3
$j = 2,\ i = 1$:	1	2	4	3
$j = 3,\ i = 1$:	1	2	4	3
$j = 3,\ i = 2$:	1	2	4	3
$j=3\ i=3$	1	2	4	3
$j=4\ i=1$	1	2	4	3
$j=4\ i=2$	1	2	4	3
$j=4\ i=3$	1	2	3	4

if given exactly upon increment:

$j=2\ i=1$	2	1	4	3
$j=3\ i=1$	1	2	4	3
$j=3\ i=2$	1	2	4	3
$j=3\ i=3$	1	2	4	3
$j=4\ i=1$	1	2	4	3
$j=4\ i=2$	1	2	4	3
$j=4\ i=3$	1	2	4	3

output: 1 2 3 4

37. Draw the "dots and arrows" representation of a relation on a nonempty finite set A that is symmetric but not transitive. (An arrow from dot a to dot b means that a is related to b. See the multiple choice for definitions of symmetric and transitive.)

38. Determine whether the following two propositions are logically equivalent:

$$\neg p \vee (q \rightarrow r), \quad q \rightarrow (\neg p \vee r)$$

p	q	r	$q \to r$	$\neg p \vee (q \to r)$	$\neg p \vee r$	$q \to (\neg p \vee r)$
T	T	T	T	T	T	T
T	T	F	F	F	F	F
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	F	T	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

same, so $\boxed{\text{Yes}}$

39. Compute the number of binary bit strings of length 8 that either begin with two 0s or end with three 1s.

$A =$ desired set

$B = \{00b_3 \cdots b_8 \mid b_3, \ldots, b_8 \in \{0,1\}\}$

$C = \{c_1 \cdots c_5 111 \mid c_1, \ldots, c_5 \in \{0,1\}\}$

$B \cap C = \{00d_3d_4d_5111 \mid d_3, d_4, d_5 \in \{0,1\}\}$

$A = B \cup C$

$|A| = |B \cup C| = |B| + |C| - |B \cap C|$

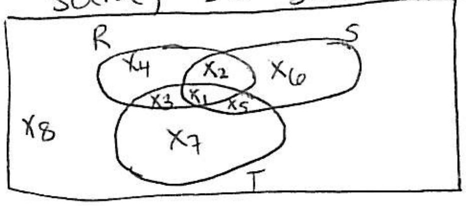
$= \boxed{2^6 + 2^5 - 2^3}$

42. For sets R, S, and T, prove that $\overline{(R \cap S \cap T)} = \overline{R} \cup \overline{S} \cup \overline{T}$. Use any of the three proof methods we discussed, but be sure to show the details. Venn diagrams only are not a proof.

Set Membership Table

	R	S	T	$R \cap S \cap T$	$\overline{R \cap S \cap T}$	$\overline{R}$	$\overline{S}$	$\overline{T}$	$\overline{R} \cup \overline{S} \cup \overline{T}$
x_1	1	1	1	1	0	0	0	0	0
x_2	1	1	0	0	1	0	0	1	1
x_3	1	0	1	0	1	0	1	0	1
x_4	1	0	0	0	1	0	1	1	1
x_5	0	1	1	0	1	1	0	0	1
x_6	0	1	0	0	1	1	0	1	1
x_7	0	0	1	0	1	1	1	0	1
x_8	0	0	0	0	1	1	1	1	1

same, so sets are equal



Set equality proof

$$\overline{R \cap S \cap T} = \{x \mid x \in \overline{R \cap S \cap T}\}$$
$$= \{x \mid x \notin R \cap S \cap T\}$$
$$= \{x \mid \neg(x \in R \wedge x \in S \wedge x \in T)\}$$
$$= \{x \mid \neg(x \in R) \vee \neg(x \in S) \vee \neg(x \in T)\}$$
$$= \{x \mid x \in \overline{R} \vee x \in \overline{S} \vee x \in \overline{T}\}$$
$$= \{x \mid x \in \overline{R} \cup \overline{S} \cup \overline{T}\}$$
$$= \overline{R} \cup \overline{S} \cup \overline{T}. \quad \square$$

43. Prove or disprove: For all integers a, b, c, if a divides bc, then a divides b or a divides c.

Disproof by counterexample.

$a = 6$, $b = 2$, $c = 3$.

$a \mid b \cdot c$ since $6 \mid 6$ but $6 \nmid 2$ and $6 \nmid 3$. $\square$

Part III. Proofs (5 pts ea.). Write complete line-by-line proofs for full credit. Substantial partial credit for good proof structure. Suggested time 35 minutes.

40. The Fibonacci numbers are defined by $f(0) = 0$, $f(1) = 1$, and for all $n \geq 2$, $f(n) = f(n-1) + f(n-2)$. Prove that for all positive integers $n \geq 2$, $f(n) \leq 2^{n-2}$.

For all integer $k \geq 2$, let $P(k)$ be the statement $f(k) \leq 2^{k-2}$

Bases $k=2$. $f(2) = f(1) + f(0) = 1 + 0 = 1 \leq 2^{2-2} = 1$. $P(2)$ True.

$k=3$. $f(3) = f(2) + f(1) = 1 + 1 = 2 \leq 2^{3-2} = 2$. $P(3)$ True.

Inductive step Let k be integer ≥ 3 and assume $P(2), \ldots, P(k)$ true.

$f(k+1) = f(k) + f(k-1)$ since $k+1 \geq 2$. (def^n $f(n)$)

$\leq 2^{k-2} + 2^{k-3}$ since $2 \leq k-1 < k$, by inductive assumption

$\leq 2^{k-2} + 2^{k-2}$ since $2^{k-3} < 2^{k-2}$

$\leq 2^{k-1}$

$\leq 2^{(k+1)-2}$

Thus $P(k+1)$ is true.

By strong induction, $P(n)$ is true for all integer $n \geq 2$. □

41. Use Mathematical Induction to prove that any positive integer amount of postage of at least 14 cents can be composed of 3 and 8 cent stamps.

For all integer $k \geq 14$, let $P(k)$ be the statement "k cents can be composed of 3 cent and 8 cent stamps."

Bases $P(14)$ true since $14¢ = 8¢ + 3¢ + 3¢$

$P(15)$ true since $15¢ = 3¢ + 3¢ + 3¢ + 3¢ + 3¢$

$P(16)$ true since $16¢ = 8¢ + 8¢$

Inductive step Let k be integer ≥ 16 and assume $P(14), \ldots, P(k)$ true.

$k+1 \geq 17$, so $k-2 \geq 14$.

$(k+1)¢ = (k-2)¢ + 3¢$

and $P(k-2)$ true by induction since $14 \leq k-2 \leq k$

Thus $P(k+1)$ true.

By strong induction, $P(n)$ true for all integers $n \geq 14$. □