I. Short answer (1.5 pts each). No partial credit – only the response will be graded. Suggested time 1 hour.

1. Suppose you want to give a proof by contrapositive of this result for all integers:
"If x is odd and y is even, then $x + y$ is odd."
Circle the letter of the assumption you would begin the proof with.

(a) x is odd and y is even. (b) x is even and y is odd. (c) $x + y$ is odd.

(d) $x + y$ is even. (e) x is even or y is odd.

2. Suppose you want to prove a theorem about the product of absolute values of real numbers $|x| \cdot |y|$. If you were to give a proof by cases, what set of cases would probably be the best to use?

(a) Both x and y nonnegative; one negative and one nonnegative; both negative

(b) Both x and y rational; one rational and one irrational; both irrational.

(c) Both x and y even; one even and one odd; both odd.

(d) $x > y$; $x < y$; $x = y$

(e) x divides y; y divides x; $\gcd(x, y) = 1$

3. According to De Morgan's laws, $\overline{A \cup (B \cap C)} =$

$\overline{A} \cap \overline{(B \cap C)} = \overline{A} \cap (\overline{B} \cup \overline{C})$

(a) $\overline{A} \cap (B \cap C)$. (b) $\overline{A} \cup (\overline{B} \cap \overline{C})$. (c) $\overline{A} \cap (\overline{B} \cup \overline{C})$.

(d) $\overline{A} \cup (B \cap C)$. (e) $\overline{A} \cap (\overline{B} \cap \overline{C})$.

4. Recall that the power set $\mathcal{P}(S)$ of the set S is defined to be the set containing all subsets of S. Let $S = \{1, 2, 3, 4\}$. In the blank to the left of each statement, circle **T** if the statement is true, and **F** if false. (Therefore this question has 5 answers!!)

T F (i) $\{\{2\}\} \subseteq \mathcal{P}(S)$ **T F** (ii) $\{1, 3\} \subseteq \mathcal{P}(S)$ **T F** (iii) $\{1, 3\} \in \mathcal{P}(S)$

T F (iv) $\{\{2\}, \{4\}\} \subseteq \mathcal{P}(S)$ **T F** (v) $\{4\} \subseteq \mathcal{P}(S)$

5. Which of these assertions is correct concerning the statement "If a^3 is irrational, then a is irrational?"

(a) This statement is false because a counterexample can be found.

(b) This statement is false because the negation of the statement can be proved easily by contradiction.

(c) This statement is true, as can be shown most easily using a direct proof.

(d) This statement is true, as can be shown most easily using a proof by contraposition.

Total 7.5

6. A standard deck of playing cards consists of 52 cards, which correspond to the set $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\} \times \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$, so that each card has one of 13 ranks and one of 4 suits. How many 5-card hands are a full house, that is, consisting of 3 cards of one rank and 2 cards of a second rank?

(a) $P(13, 2) \cdot P(4, 3) \cdot P(4, 2)$ (b) $13 \cdot 12 \cdot P(4, 3) \cdot P(4, 2)$

(c) $13 \cdot 12 \cdot C(4, 3) \cdot C(4, 2)$ (d) $C(13, 2) \cdot C(4, 3) \cdot C(4, 2)$

product rule of counting

pick triple rank: 13 ways
pick pair rank: 12 ways
pick triple suits C(4,3) ways
pick pair suits C(4,2) ways

7. How many permutations of the letters in the word SASSAFRAS are there?

(a) $C(9, 4) \cdot C(9, 3)$ (b) $9!/(4! \cdot 3!)$ (c) $9!/(4! + 3!)$

(d) $P(9, 4) \cdot P(9, 3)$ (e) $9!/7$

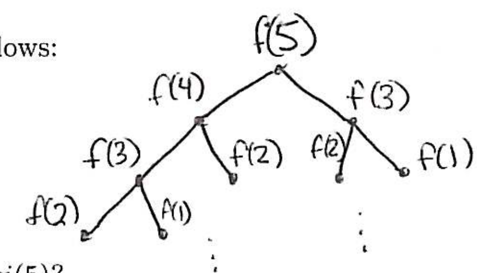
9 letters S:4 A:3 others:1

Permutations with repetition

8. What is the best big-oh notation for the number of comparisons used by mergesort to sort a list of n numbers?

Answer: $O(n \log n)$

9. A recursive algorithm for computing the Fibonacci numbers is as follows:



procedure *fibonacci*(n : nonnegative integer)
if $n = 0$ **then** return 0
else if $n = 1$ **then** return 1
else return $fibonacci(n - 1) + fibonacci(n - 2)$

How many times is $fibonacci(2)$ called in order to compute $fibonacci(5)$?

(a) 1 (b) 2

(c) 3 (d) 5

(e) 8 (f) 13

10. How many different license plates are available if a licence plate consists of 3 decimal digits (from $\{0, 1, \ldots, 9\}$) followed by 4 uppercase letters?

(a) $P(10, 3) \cdot P(26, 4)$ (b) $C(10, 3) \cdot C(26, 4)$ (c) $10^3 \cdot 26^4$

(d) $3 \cdot 10 + 4 \cdot 26$ (e) $10^3 + 26^4$ (f) $P(10, 3) + P(26, 4)$

3-permutation of $\{0, .., 9\}$ with repeats allowed 10^3
then 4-permutation of $\{A, .., Z\}$ with repeats allowed $\cdot\ 26^4$

Total 7.5

11. Suppose $f : \mathbb{R} \to \mathbb{R}$ has the following property for all real numbers x and y: if $x < y$ then $f(x) < f(y)$. (I.e., f is strictly increasing.) Which of the following is true?

   (a) f must be both 1-1 and onto $\mathbb{R}$.

   (b) f is not necessarily 1-1 and not necessarily onto $\mathbb{R}$.

   (c) f must be 1-1 but is not necessarily onto $\mathbb{R}$.

   (d) f is onto $\mathbb{R}$ but is not necessarily 1-1.

12. Write down the initial term and common ratio of the series $\sum_{i=2}^{\infty} \frac{3 \cdot 5^i}{4^i}$.

   initial term: $\frac{3 \cdot 5^2}{4^2}$ common ratio: $\frac{5}{4}$

   $\frac{3 \cdot 5^2}{4^2} + \frac{3 \cdot 5^3}{4^3} + \cdots$

13. Write down the initial term and common difference of the progression $(6, -3, -12, -21, \ldots)$.

   initial term: 6 common difference: -9

14. Suppose $f : \mathbb{Z} \to \mathbb{R}$ has the rule $f(n) = 2n - 4$. Circle the letter corresponding to the range of f.

   (a) the set of even integers **(b)** the real numbers **(c)** the rational numbers

   (d) the set of natural numbers $\{0, 1, 2, \ldots\}$ **(e)** $\mathbb{Z}$ **(f)** the set of odd integers

15. Find functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $g \circ h$ has the rule $(g \circ h)(x) = \lfloor x^2 + 7 \rfloor$.

   (a) $h(x) = \lfloor x \rfloor$, $g(x) = x^2 + 7$. **(b)** $h(x) = x^2 + 7$, $g(x) = \lfloor x \rfloor$.

   (c) $h(x) = x^2$, $g(x) = \lfloor x \rfloor + 7$. **(d)** $h(x) = \lfloor x \rfloor + 7$, $g(x) = x^2$.

   (e) $h(x) = x + 7$, $g(x) = \lfloor x^2 \rfloor$. **(f)** $h(x) = \lfloor x + 7 \rfloor$, $g(x) = x^2$.

16. Circle the letter for the correct statement about the function $g : \mathbb{Z} \to \mathbb{Z}$ with rule $g(x) = 2x$.

   (a) g is one-to-one but not onto. **(b)** g is neither one-to-one nor onto.

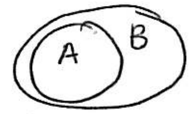
   (c) g is one-to-one and onto. **(d)** g is onto but not one-to-one.

Total 9

17. Circle the letter of the statement which makes the implication about sets true. If $A \subseteq B$, then

(a) $\overline{B} \subseteq \overline{A}$. (b) $B - A \subseteq A - B$. (c) $A \cup B = A \cap B$.

(d) $B \subseteq A \cap B$. (e) $B - A \neq \emptyset$.



18. Circle the letter of the inverse of the statement "If $\overbrace{\text{it is dark outside}}^{p}$, then $\overbrace{\text{I stay at home}}^{q}$."

(a) If I stay at home, then it is dark outside.

(b) It is dark outside and I do not stay at home.

(c) If I do not stay at home, then it is not dark outside.

(d) If it is not dark outside, then I do not stay at home.

form: $p \to q$

inverse: $\neg p \to \neg q$

19. Assume that x, y, and z are all real numbers. Circle the letter of the negation of the statement "x is positive, and y and z are negative."

(a) If $x \leq 0$, then $y \geq 0$ or $z \geq 0$.

(b) $x \leq 0$ or $y \geq 0$ or $z \geq 0$.

(c) x is negative and y and z are positive.

(d) $x \leq 0$, and either $y \geq 0$ or $z \geq 0$.

(e) Either $x \leq 0$, or $y \geq 0$ and $z \geq 0$.

$\neg(p \wedge q \wedge r) \equiv \neg p \vee \neg q \vee \neg r$

p: $x > 0$

q: $y < 0$

r: $z < 0$

20. Suppose $P(x, y, z)$ is a predicate where the universe for x, y, and z is $\{1, 2\}$. Also suppose that the predicate is true in the following cases – $P(1,1,1)$, $P(1,1,2)$, $P(2,1,1)$, $P(2,2,2)$ – and false otherwise. Circle the letter of the true quantified statement.

(a) $\exists x \forall y \forall z \; P(x, y, z)$

(b) $\forall x \exists z \forall y \; P(x, y, z)$

(c) $\exists x \exists y \forall z \; P(x, y, z)$ $x=1$, $y=1$

(d) $\exists x \exists z \forall y \; P(x, y, z)$

21. Circle the letter of the negation of $\forall a \exists b \; (a > b)$.

(a) $\forall a \exists b \; (a \leq b)$ (b) $\exists a \forall b \; (a > b)$ (c) $\exists a \exists b \; (a \leq b)$ (d) $\exists a \forall b \; (a \leq b)$

Total 7.5

22. Using the Binomial Theorem, find the sum of the series $\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j}$.

(a) -1 (b) 0

(c) 1 (d) 2^n

(e) $2^n/2$ (f) 21

$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$

$x = -1 \quad y = 1$

23. What is the probability of rolling a multiple of 3 on a fair 6-sided die?

(a) 1 (b) 5/6 (c) 4/6

(d) 3/6 (e) 2/6 (f) 1/6

24. (2pts) Recall the following properties of a relation $R \subseteq A \times A$ (saying $a\,R\,b$ is the same as saying $(a, b) \in R$:

Reflexive:	For all $a \in A$, $a\,R\,a$.
Symmetric:	For all $a, b \in A$, if $a\,R\,b$, then $b\,R\,a$.
Antisymmetric:	For all $a, b \in A$, if $a\,R\,b$ and $b\,R\,a$, then $a = b$.
Transitive:	For all $a, b, c \in A$, if $a\,R\,b$ and $b\,R\,c$, then $a\,R\,c$.

Check the boxes to the left of the properties satisfied by the relation $R_{\leq} = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$.

- [x] reflexive
- [ ] symmetric
- [x] antisymmetric
- [x] transitive

25. A fast food restaurant offers 3 types of items: tacos, hamburgers, and chicken. Assume there is an unlimited supply of each type of item. In how many ways can a very hungry person buy 5 items if the order of the items does not matter?

(a) $C(5, 3)$ (b) $P(5, 3)$

(c) $C(7, 2)$ (d) $P(7, 2)$

(e) 5^3

stars and bars / balls and bins

tacos | ham burgers | chicken

00000

$C(5+3-1, 3-1)$

26. What is the minimum number of persons in a group necessary to guarantee that at least 3 of them were born on the same day of the week?

(a) 9 (b) 4

(c) 16 (d) 15

(e) 14 (f) 21

days of week = 7 = n

3 = k

$N = 2 \cdot n + 1$

$= (k-1) \cdot n + 1 = 2 \cdot 7 + 1 = 15$

Total 8

27. Compute $\gcd(2^3 \cdot 3^2 \cdot 5^2, 2^5 \cdot 3^4 \cdot 7^4)$.

Answer: $2^3\, 3^2$

28. Compute the following: $-32 \mod 7 =$ 3 $\qquad$ $30 \mod 13 =$ 4

$-32 = -5 \cdot 7 + 3$ $\qquad$ $30 = 2 \cdot 13 + 4$

29. Suppose that $P(n)$ is the statement "$n+1 = n+2$." What is wrong with the following "proof" that the statement $P(n)$ is true for all nonnegative integers n:

You assume that $P(k)$ is true for some positive integer k; that is, $k+1 = k+2$. Then you add 1 to both sides of this equation to obtain $k+2 = k+3$; therefore $P(k+1)$ is true. By the principle of mathematical induction $P(n)$ is true for all nonnegative integers n.

(a) There is nothing wrong with this proof.

(b) The proof is incorrect because the statement used in the inductive hypothesis is incorrect.

(c) The proof is incorrect because there is no basis step.

(d) The proof is incorrect because you cannot add 1 to both sides of the equation in the inductive step.

30. For which of the following is the recursively defined set S equal to the set of odd positive integers?

(a) $2 \in S$; $x \in S \rightarrow x+2 \in S$ $\qquad$ (b) $1 \in S$; $x \in S \rightarrow 2x+1 \in S$

(c) $1 \in S$; $3 \in S$; $x \in S \rightarrow x+4 \in S$. $\qquad$ (d) $99 \in S$; $x \in S \rightarrow x-2 \in S$

(e) None of these

31. An algorithm finds the maximum number in a list of n numbers. What is a reasonable operation with which to measure the complexity of the algorithm, and what is the best big-oh notation for the number of these operations assuming the algorithm is efficient?

operation: Comparison $\qquad$ big-oh complexity: $O(n)$

32. Circle the letter corresponding to the true statement.

(a) If $f(x)$ is $O(g(x))$ then $g(x)$ is $O(f(x))$. $\qquad$ (b) If $f(x)$ is $\Omega(g(x))$ then $g(x)$ is $\Theta(f(x))$.

(c) If $f(x)$ is $\Omega(g(x))$ then $g(x)$ is $\Omega(f(x))$. $\qquad$ (d) If $f(x)$ is $\Theta(g(x))$ then $g(x)$ is $\Theta(f(x))$.

33. Is 143 prime? (Circle one.) $\quad 2 \nmid 143 \quad 3\overline{)143}$ 4, 12, 23 $\quad 5 \nmid 143 \quad 7 \nmid 143 \quad 11\overline{)143}$ 13, 11, 33

YES $\quad$ **NO**

Total 10.5

Part II. Computation, Algorithms, and Examples (5 pts ea.). Show work for full credit. Suggested time 25 minutes.

34. Compute the number of binary bit strings of length 7 that either begin with three 1s or end with two 0s.

$B = \{111 b_4 b_5 b_6 b_7 : b_i \in \{0,1\} \text{ for } i=4,5,6,7\}$

$C = \{c_1 c_2 c_3 c_4 c_5 00 : c_i \in \{0,1\} \text{ for } i=1,2,3,4,5\}$

Answer $= |B \cup C|$.

$|B| = 2^4$

$|C| = 2^5$

$B \cap C = \{111 d_4 d_5 00 : d_4, d_5 \in \{0,1\}\}$

$|B \cap C| = 2^2$.

Inclusion-exclusion: $|B \cup C| = |B| + |C| - |B \cap C| = \boxed{2^4 + 2^5 - 2^2}$

35. Trace through the Insertion Sort algorithm on the list $2, 4, 1, 3$ by writing down the order of the list after each increment of j or i.

procedure insertionSort($a_1, \ldots, a_n$: reals with $n \geq 2$)
for $j := 2$ **to** n
begin
  $i := 1$
  while $a_j > a_i$
    $i := i + 1$
  $m := a_j$
  for $k := 0$ **to** $j - i - 1$
    $a_{j-k} := a_{j-k-1}$
  $a_i := m$
end

Trace

initial list:	2	4	1	3
$j = 2,\ i = 1$:	2	4	1	3
$j = 2,\ i = 2$:	2	4	1	3
$j = 3,\ i = 1$:	1	2	4	3
$j = 4,\ i = 1$:	1	2	4	3
$j = 4,\ i = 2$:	1	2	4	3
$j = 4,\ i = 3$:	1	2	3	4

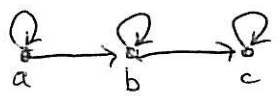
list written just before next change in i or j.

initial:	2	4	1	3
$j=2, i=1$:	2	4	1	3
$j=2, i=2$:	2	4	1	3
$j=3, i=1$:	2	4	1	3
$j=4, i=1$:	1	2	4	3
$j=4, i=2$:	1	2	4	3
$j=4, i=3$:	1	2	4	3
exit :	1	2	3	4

← if list is written just after the increment, see left (not preferred)

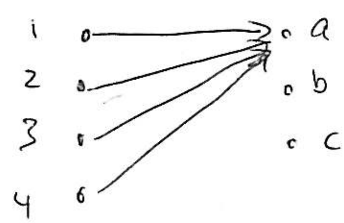
Total 10

36. Draw the "dots and arrows" representation of a relation on a nonempty finite set A that is reflexive but not transitive. (An arrow from dot a to dot b means that a is related to b. See the multiple choice for definitions of reflexive and transitive.)



many correct answers

37. Give an example of a function with domain $\{1, 2, 3, 4\}$ and codomain $\{a, b, c\}$ which is not one-to-one and not onto.



Total 10

38. Use the Euclidean algorithm to find $\gcd(126, 110)$.

$$126 = 1 \cdot 110 + 16$$
$$110 = 6 \cdot 16 + 14$$
$$16 = 1 \cdot 14 + \boxed{2}$$
$$14 = 7 \cdot 2 + 0$$

$\gcd = 2$

39. Determine whether the following two propositions are logically equivalent:

$p \to (\neg q \wedge r), \quad \neg p \vee \neg(r \to q)$

p	q	r	$\neg q \wedge r$	$p \to (\neg q \wedge r)$	$\neg(r \to q)$	$\neg p \vee \neg(r \to q)$
T	T	T	F	F	F	F
T	T	F	F	F	F	F
T	F	T	T	T	T	T
T	F	F	F	F	F	F
F	T	T	F	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	T	T
F	F	F	F	T	F	T

Same, so $\boxed{\text{yes}}$

Total 10

Part III. Proofs (5 pts ea.). Write complete line-by-line proofs for full credit. Substantial partial credit for good proof structure. Suggested time 35 minutes.

40. Use Mathematical Induction to prove that any positive integer amount of postage of at least 12 cents can be composed of 3 and 7 cent stamps.

For $k \in \mathbb{Z}^+$, let $P(k)$ be the statement "k cents in postage can be composed of 3 cent and 7 cent stamps.

Bases $P(12)$ True, since $12¢ = 3¢ + 3¢ + 3¢ + 3¢$
$P(13)$ True, since $13¢ = 7¢ + 3¢ + 3¢$
$P(14)$ True, since $14¢ = 7¢ + 7¢$.

Inductive step Let $k \geq 14$, and assume $P(12), \ldots, P(k)$ true.

Since $14 \leq k$,
$15 \leq k+1$ and $12 \leq k-2 \leq k$

$(k+1)¢ = (k-2)¢ + 3¢$,
and $12 \leq k-2 \leq k$ implies that $(k-2)¢$ can be composed of $3¢$ and $7¢$ stamps by inductive hypothesis.
Therefore $P(k+1)$ is true.

By strong induction, $P(k)$ is true for all integer $k \geq 12$. □

41. Prove or disprove: For all integers r, s, t, u, if r divides s and t divides u, then $(r+s)$ divides $(t+u)$.

Disproof by counterexample.

$r = 1,\ s = 1,\ t = 1,\ u = 2$.

$1 \mid 1$ and $1 \mid 2$, but $(1+1) \nmid (1+2)$. □

42. The Fibonacci numbers are defined by $f(0) = 0$, $f(1) = 1$, and for all $n \geq 2$, $f(n) = f(n-1) + f(n-2)$. Prove that for all positive integers $n \geq 2$, $f(n) \leq 2^{n-2}$.

For all integer $k \geq 2$, let $P(k)$ be the statement $f(k) \leq 2^{k-2}$.

Bases Let $k=2$. $f(2) = f(1) + f(0)$ by definition fibonacci #

$= 1 + 0 = 1$.

$1 \leq 2^{2-2} = 1$. So $P(2)$ is true.

$k=3$: $f(3) = f(2) + f(1) = 1+1 = 2 \leq 2^{3-2} = 2$. So $P(3)$ is true.

Inductive step Let k be an integer ≥ 3 and assume $P(2), \ldots, P(k)$ true.

$f(k+1) = f(k) + f(k-1)$, since $k+1 \geq 2$.

$\leq 2^{k-2} + 2^{k-3}$ since $P(k), P(k-1)$ true.

$\leq 2^{k-2} + 2^{k-2}$ since $2^{k-3} < 2^{k-2}$

$f(k+1) \leq 2^{k-1} = 2^{(k+1)-2}$

Therefore $P(k+1)$ is true.

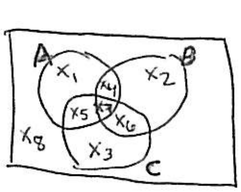
By strong induction, $P(n)$ is true for all integer $n \geq 2$. □

43. For sets A, B, and C, prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Use any of the three proof methods we discussed, but be sure to show the details. Venn diagrams only are not a proof.

By set membership table

	A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
x_1	X							
x_2		X		X				
x_3			X	X				
x_4	X	X		X	X	X		X
x_5	X		X	X	X		X	X
x_6		X	X	X				
x_7	X	X	X	X	X	X	X	X
x_8								

same for each possible type of element; therefore $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □



Set equality directly

$A \cap (B \cup C) = \{x \mid x \in A \cap (B \cup C)\}$

$= \{x \mid x \in A \wedge x \in B \cup C\}$

$= \{x \mid x \in A \wedge (x \in B \vee x \in C)\}$

$= \{x \mid (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\}$

distributive law $\wedge, \vee$

$= \{x \mid x \in A \cap B \vee x \in A \cap C\}$

$= \{x \mid x \in (A \cap B) \cup (A \cap C)\}$

$= (A \cap B) \cup (A \cap C)$. □