Practice Exam 2

On Exam 2, show work for full credit. Partial credit for good proof structure even if proof is not correct.

Find a formula that generates the following sequence $a_1, a_2, a_3, \ldots$.

1. $5, 9, 13, 17, 21, \ldots$.
2. $1, 1/3, 1/5, 1/7, 1/9, \ldots$.
3. $0, 2, 0, 2, 0, 2, 0, \ldots$.

4. Find the sum $1 - 1/2 + 1/4 - 1/8 + 1/16 - \ldots$.
5. Find the sum $112 + 113 + 114 + ... + 673$.
6. Find $\sum_{j=1}^{3} \sum_{i=1}^{j} ij$.
7. Rewrite $\sum_{i=-3}^{4} (i^2 + 1)$ so that the index of summation has lower limit 0 and upper limit 7.
8. Find $\sum_{j=1}^{100} (2j + 5 + 3^j)$.
9. Write pseudocode for an algorithm that takes a list of n integers $a_1, a_2, \ldots, a_n$ and finds the number of integers each greater than five in the list.
10. Write pseudocode for an algorithm that takes a list of n integers ($n \geq 1$) and finds the average of the largest and smallest integers in the list.
11. Describe how the binary search algorithm searches for 27 in the following list: 5 6 8 12 15 21 25 31.
12. Prove that $1^2 + 2^2 + \ldots + n^2$ is $O(n^3)$. (Technically speaking this requires induction, but you can skip any inductive proof.)
13. Find witnesses C and k from the definition of big-oh to show that $f(n) = 3n^2 + 8n + 7$ is $O(n^2)$. (Hint: $x > k \to |f(n)| \leq C|n^2|$.)
14. Find the best big-O function for $\dfrac{x^3 + 7x}{3x + 1}$.

In the questions below find the best big-oh notation to describe the complexity of the algorithm.

15. A binary search of n elements.
16. A linear search to find the smallest number in a list of n numbers.
17. An algorithm that lists all ways to put the numbers $1, 2, 3, \ldots, n$
18. An algorithm that prints all bit strings of length n.

In the questions below find the best big-oh notation to describe the number designated steps of the algorithm.

19. The number of print statements in the following:

$i := 1$
$j := 1$
while $i \leq n$
begin
    while $j \leq i$
    begin
        print "hello"
        $j := j + 1$
    end
    $i := i + 1$
end

20. The number of print statements in the following:

while $n > 1$
begin
    print "hello"
    $n := \lfloor n/2 \rfloor$
end

20. The number of comparisons in the best case, average case, and worst case analysis of linear search (p.170).

21. Prove or disprove: For all integers a, b, c, d, if $a|b$ and $c|d$, then $(a + c)|(b + d)$.

22. Prove or disprove: For all integers a, b, c, if $a|b$ and $b|c$ then $a|c$.

23. Prove or disprove: For all integers a, b, c, if $a|bc$, then $a|b$ or $a|c$.

24. Prove or disprove: For all integers a, b, c, if $a|c$ and $b|c$, then $ab|c^2$.

Use the improved algorithm discussed in class to find the prime factorizations of the following.

25. Find the prime factorization of 510,510.

26. Find the prime factorization of 45,617.

27. List all positive integers less than 30 that are relatively prime to 20.

28. Find $\gcd(20!, 12!)$.

29. Find $\gcd(2^{89}, 2^{346})$.

30. Find $\text{lcm}(20!, 12!)$.

31. Find $\text{lcm}(2^{89}, 2^{346})$.

32. Suppose that the $\text{lcm}(a, b) = 400$ and $\gcd(a, b) = 10$. If $a = 50$, find b.

33. Applying the division algorithm with $a = -41$ and $d = 6$ yields what value of r?

34. Find 18 mod 7.

35. Find 88 mod 13.

36. Find the hexadecimal expansion of $ABC_{16} + 2F5_{16}$.

37. Prove or disprove: The sum of two primes is a prime.

38. Prove or disprove: If p and q are primes, both > 2, then $p + q$ is composite.

39. Find the smallest positive integer a such that $a + 1 \equiv 2a \pmod{11}$.

40. Find four integers b (two negative and two positive) such that $7 \equiv b \pmod 4$.

41. Prove or disprove. Let a, b, c, d, and m be integers with $m > 1$. If $a \equiv b \pmod m$ and $c \equiv d \pmod m$, then $ac \equiv b + d \pmod m$.

42. Prove or disprove. Let a, b, c, d, and m be integers with $m > 1$. If $a \equiv b \pmod m$, then $2a \equiv 2b \pmod m$.

43. Encrypt the message *NEED HELP* by translating the letters into numbers, applying the encryption function $f(p) = (3p + 7) \bmod 26$, and then translating the numbers back into letters.

44. A message has been encrypted using the function $f(x) = (x+5) \bmod 26$. If the message in coded form is *JCFHY*, decode the message.

45. Use the Euclidean algorithm to find $\gcd(44, 52)$.

46. Use the Euclidean algorithm to find $\gcd(900, 140)$.

47. Suppose you wish to use the Principle of Mathematical Induction to prove that $11! + 22! + 33! + \ldots + nn! = (n + 1)! - 1$ for all $n \geq 1$.
    (a) Write $P(1)$
    (b) Write $P(5)$
    (c) Write $P(k)$
    (d) Write $P(k + 1)$
    (e) Use the Principle of Mathematical Induction to prove that $P(n)$ is true for all $n \geq 1$.

48. Use the Principle of Mathematical Induction to prove that $1 + 3 + 9 + 27 + \cdots + 3^n = \frac{3^{n+1}-1}{2}$ for all $n \geq 0$.

49. Use the Principle of Mathematical Induction to prove that $2n + 3 \leq 2^n$ for all $n \geq 4$.

50. Use the Principle of Mathematical Induction to prove that $3 | (n^3 + 3n^2 + 2n)$ for all $n \geq 1$.

51. (Tower of Hanoi puzzle) Let n be a positive integer. The tower of Hanoi puzzle has three pegs A, B, and C. The starting position is n disks on peg A, where every disk is strictly smaller than the one underneath. How many moves does it take to transfer all disks to peg C with the following restrictions: (i) only one disk may be moved at a time, and (ii) a disk may never lie on top of a smaller disk? Conjecture and prove a formula. The following diagram is for $n = 4$.

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    |          |            |
    =          |            |
   ==          |            |
  ===          |            |
  ====         |            |
 base A      base B       base C
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