I. Short answer (3pts each). Do not give proofs.

1. **TRUE** **FALSE** (Circle one). In the US coin system, which consists of 1, 5, 10, 25, and 50 cent coins, the greedy way of making change is also the optimal way (when plenty of all coins are available).

2. **TRUE** **FALSE** (Circle one). Insertion sort uses $O(n^3)$ comparisons to sort a list of length n.

   $O(n^2)$

3. **YES** **NO** (Circle one). Is 131 prime?

   try $2, 3, 5, 7, 11 = \lfloor\sqrt{131}\rfloor$

4. In the best big-oh notation, what is the number of comparisons needed in an iterative algorithm that computes the maximum element of a list of n integers? $O(n)$

5. In the best big-oh notation, what is the number of comparisons needed in the binary search algorithm which searches for a given x in a sorted list of n numbers? $O(\log n)$

6. Name an algorithm that we studied for which the best-case time complexity was different from the worst-case time complexity. linear search

7. Compute: $\gcd(12, 18) =$ 6 $\gcd(2^5, 2^8) =$ 2^5 $\gcd(5, 0) =$ 5

   $\gcd(2^2 \cdot 3, 2 \cdot 3^2)$

8. Compute: $\text{lcm}(11, 13) =$ 143 $\text{lcm}(2^2 5^5, 2^5 5^2) =$ $2^5 5^5$ $\text{lcm}(12, 18) =$ 36

   $\text{lcm}(2^2 \cdot 3, 2 \cdot 3^2)$

9. For the arithmetic progression $3\pi/4, 5\pi/4, 7\pi/4, 9\pi/4, \ldots$, give the
   common difference: $\pi/2$

   $\frac{5\pi}{4} - \frac{3\pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$

10. For the geometric progression $2, -1/2, 1/8, -1/32, \ldots$, give the
    initial term: 2 and the
    common ratio: $-1/4$

    $r = \frac{-1/2}{2} = \frac{-1}{4}$

11. Compute: $-9 \bmod 7 =$ 5 $\qquad$ $25 \bmod 7 =$ 4

    $-9 = -2 \cdot 7 + 5$ $\qquad$ $25 = 3 \cdot 7 + 4$

12. Compute: $A9_{16} + 3C_{16} =$ $E5_{16}$ $\qquad$ $10010_2 + 1011_2 =$ 11101_2

    $$\begin{array}{r} {}^{1}A9 \\ 3C \\ \hline E5 \end{array} \qquad \begin{array}{r} 10010 \\ 1011 \\ \hline 11101 \end{array}$$

II. Computation Problems 13-16 (10pts each). Show work to clearly justify your answer.

13. Use the Euclidean algorithm to compute $\gcd(116, 76)$.

$$116 = 1 \cdot 76 + 40$$
$$76 = 1 \cdot 40 + 36$$
$$40 = 1 \cdot 36 + \boxed{4}$$
$$36 = 9 \cdot 4 + 0$$

$$\boxed{\gcd(116, 76) = 4}$$

14. Trace through the Bubble Sort algorithm on the list 3, 1, 4, 2 by writing down the order of the list after each increment of i or j.

procedure bubblesort($a_1, \ldots, a_n$: real with $n \geq 2$)
for $i := 1$ **to** $n-1$
  for $j := 1$ **to** $n-i$
    if $a_j > a_{j+1}$ **then** interchange a_j and a_{j+1}

Trace

initial list:	3	1	4	2
$i = 1, j = 1$:	1	3	4	2
$i = 1, j = 2$:	1	3	4	2
$i=1, j=3$	1	3	2	4
$i=2, j=1$	1	3	2	4
$i=2, j=2$	1	2	3	4
$i=3, j=1$	1	2	3	4

15. Find $\sum_{j=1}^{50} 2j$ and $\sum_{i=2}^{10} 4^i$.

$$\sum_{j=1}^{50} 2j = 2\sum_{i=1}^{50} i = \boxed{2 \cdot \frac{50(51)}{2} = 2550}$$

$$\sum_{i=2}^{10} 4^i = \boxed{\frac{4^2 - 4^{11}}{1-4}} \quad \left(\frac{a_2 - a_{11}}{1-r} \right)$$

16. Let $f(n) = 3n^2 + 8n + 10$. Give witnesses C and k that show $f(n)$ is $O(n^2)$. (Hint: $n > k \rightarrow |f(n)| \leq C|n^2|$.)

set $\boxed{k=1.}$ When $n > k$,

$$\begin{aligned} 3n^2 &\leq 3n^2 \\ 8n &\leq 8n^2 \\ 10 &\leq 10n^2 \\ \hline & \ \ 21n^2 \end{aligned}$$

so $\boxed{C=21}$

$$\left(|3n^2+8n+10| \leq |3n^2|+|8n|+|10| \underset{\substack{\uparrow \\ n>1=k}}{\leq} 3n^2+8n^2+10n^2 = 21|n^2| \right)$$

III. Proofs (12pts each). Partial credit for good proof structure.

17. (Recall that by definition, if a and b are integers and m is a positive integer, then $a \equiv b \pmod{m}$ provided that m divides $a-b$.) Prove the following. Let a, b, c be integers, and let m be a positive integer. Show that if $a \equiv b \pmod{m}$, then $a - c \equiv b - c \pmod{m}$.

Proof (direct)

Let $a, b, c \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$.

Assume $a \equiv b \pmod{m}$.

Then $m \mid (a-b)$ by definition of $\equiv$.

Then $\exists d \in \mathbb{Z}$ such that $m \cdot d = a - b$ by definition of divides (1).

Thus $m \cdot d = (a-c) - (b-c)$ by adding + subtracting c.

By definition, ~~m·d~~ $m \mid [(a-c)-(b-c)]$,

and by definition, $a - c \equiv b - c \pmod{m}$ $\square$

18. Prove that $2^n < n!$ for all integers $n \geq 4$.

Proof (by induction) For $k \in \mathbb{Z}$,

Let $P(k)$ be the statement $2^k < k!$.

Basis Fix $k = 4$.

$2^k = 2^4 = 16$ $k! = 4! = 24$, so $P(4)$ true.

Inductive step Let k be an integer with $k \geq 4$.

Assume $P(k)$ true; i.e.,

$2^k < k!$

also $2 < k+1$ since $k+1 \geq 5$.

Thus $2^{k+1} < (k+1)!$,

and so $P(k+1)$ is true.

By the Principle of Mathematical Induction, for all integer $n \geq 4$, $P(n)$ true. $\square$