PRINT Last name: Key B First name:

Signature: Student ID:

Math 230 Exam 1, Spring 2008

Show work for full credit. Partial credit for good proof structure even if proof is not correct.

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1. A couple of weeks ago, I used the logical equivalence $p \to (q \vee r) \equiv (p \wedge \neg q) \to r$ to simplify a proof in a paper. Prove that this actually is a logical equivalence.

Way ① $p \to (q \vee r)$ is False exactly when p is true and $q \vee r$ is False, or p True, q False, r False.

$(p \wedge \neg q) \to r$ is False exactly when $p \wedge \neg q$ true, r False, or p true, q False, r False.

Same

Way ②

p	q	r	$q \vee r$	$p \to (q \vee r)$	$(p \wedge \neg q)$	$(p \wedge \neg q) \to r$
T	T	T	T	T	F	T
T	T	F	T	T	F	T
T	F	T	T	T	T	T
T	F	F	F	F	T	F
F	T	T	T	T	F	T
F	T	F	T	T	F	T
F	F	T	T	T	F	T
F	F	F	F	T	F	T

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2. Recall that a logical proposition involving only propositional variables, negations, and conjunctions (e.g., $p \wedge q \wedge \neg r$) is true for only one row of its truth table. Using this as a building block, find a proposition having the following truth table.

p	q	r	??
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

$(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$

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3. Write the contrapositive, converse, and inverse of the following: You sleep late if it is Saturday.

if (it is Saturday) then (you sleep late)

Contrapositive: If you don't sleep late then it is not Saturday

Converse: If you sleep late then it is Saturday

Inverse: If it is not Saturday then you do not sleep late.

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4. Circle the negation of the proposition $\forall x\big((P(x) \vee Q(x)) \to \neg R(x)\big)$.

negation changes to $\exists x$

$\neg(p \to q) \equiv p \wedge \neg q$

~~(a)~~ $\forall x((\neg P(x) \wedge \neg Q(x)) \vee \neg R(x))$

(b) $\exists x((P(x) \vee Q(x)) \wedge R(x))$

~~(c)~~ $\forall x(R(x) \to (\neg P(x) \wedge \neg Q(x)))$

(d) $\exists x(\neg P(x) \vee \neg Q(x) \vee \neg R(x))$

~~(e)~~ $\forall x(\neg R(x) \to (P(x) \vee Q(x)))$

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5. Write $\mathbb{T}$ next to the tautology. Write $\mathbb{F}$ next to the contradiction. Do nothing for the rest of the propositions.

____(a) $(p \oplus q) \wedge \neg p$

F (b) $(p \leftrightarrow q) \wedge (p \oplus q)$ since $\neg(p \Leftrightarrow q) \equiv p \oplus q$

T (c) $p \vee (q \vee \neg p)$ p true $\Rightarrow$ true p false $\Rightarrow$ true

____(d) $p \wedge (r \vee p)$ ~~p true~~

____(e) $q \rightarrow \neg q$

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6. Suppose the variable x represents students, y represents courses, and $T(x, y)$ means "x is taking y". Next to the quantified statement, write the number of the equivalent English sentence.

4 $\forall y \exists x\, T(x, y)$	**(1)**	Some student is taking every course.
1 $\exists x \forall y\, T(x, y)$	**(2)**	No student is taking any course.
2 $\neg(\exists x \exists y\, T(x, y))$	**(3)**	Some students are taking no courses.
3 $\neg(\forall x \neg(\forall y\, \neg T(x, y)))$	**(4)**	Every course is being taken by at least one student.

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7. Explain the case for truth values of p and q that shows that the following argument is not valid.

$$\begin{array}{l} p \rightarrow q \\ \neg p \\ \hline \therefore \neg q \end{array}$$

p false, q true make hypotheses

$p \rightarrow q$, $F \Rightarrow T$, which is True

$\neg p$, T also true

but conclusion $\neg q$ is False.

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8. Prove the following statement. If n^3 is an odd integer then n is odd. (Hint: is direct, contrapositive or contradiction easier?)

Proof (contrapositive)

Let n be an integer.

Assume n is even.

Then $n = 2k$ for some $k \in \mathbb{Z}$.

$n^3 = (2k)^3 = 8k^3 = 2(4k^3)$

so n^3 is even $\square$

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9. Prove that the equation $x^2 + 3y^2 = 11$ has no integer solutions.

Proof by exhaustive cases. Since $0 \le x^2 \le 11$, forces $x = 0, \pm 1, \pm 2, \pm 3$.
since $0 \le 3y^2 \le 11$, forces $y = 0, \pm 1$,
Since squaring removes the minus sign, we need check only 8 cases.

$y \backslash x$	0	1	2	3
0	0	1	4	9
1	3	4	7	12

Therefore no integer solutions.

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10. Recall that by definition $A \oplus B = (A - B) \cup (B - A)$. Prove by set membership table that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.

A	B	C	$B \oplus C$	$A \oplus (B \oplus C)$	$A \oplus B$	$(A \oplus B) \oplus C$
1	1	1	0	1	0	1
1	1	0	1	0	0	0
1	0	1	1	0	1	0
1	0	0	0	1	1	1
0	1	1	0	0	1	0
0	1	0	1	1	1	1
0	0	1	1	1	0	1
0	0	0	0	0	0	0

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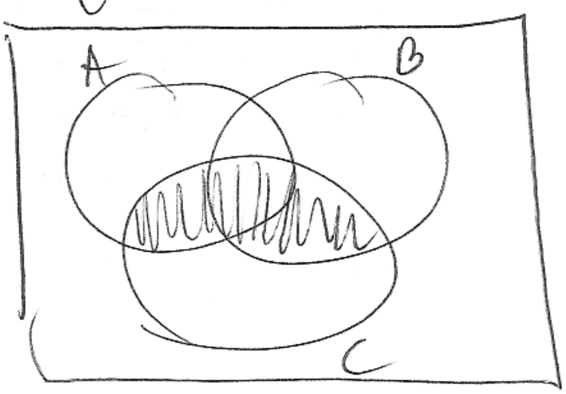
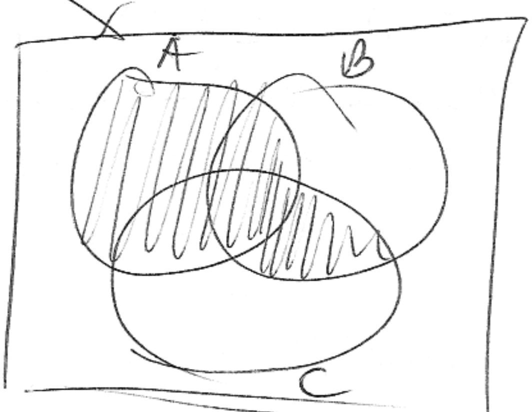
11. Draw two Venn diagrams (one for the left side, one for the right side) to justify which relationship, $\subseteq$, $=$, or $\supseteq$, is valid for the following pair of sets. Write the correct operator in the blank.

$(A \cup B) \cap C$ $\underline{\ \subseteq\ }$ $A \cup (B \cap C)$





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12. In specifying a function, give the domain and codomain. Also give either the rule or clearly describe how the function is defined.

(a) Give an example of a function which is strictly increasing.

(b) Give an example of a function which is (monotonic) increasing but not strictly increasing.

(a) $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x$

-or-

$f: \mathbb{R} \to \mathbb{R}$
$f(x) = e^x$

many solutions

(b) $f: \mathbb{R} \to \mathbb{R}$
$f(x) = 1$

-or-

$f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0, & x \le 0 \\ x, & x > 0 \end{cases}$$

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13. In the space provided, write whether or not the rule with the given domain and codomain describes a function, and if not give a brief reason why. Recall the natural numbers are $\mathbb{N} = \{0, 1, 2, \ldots\}$.

(c) $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ equals any integer greater than n.

No (Yes/No) Reason: f(n) has more than one possible value

(b) $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = \sqrt{x}$.

Yes (Yes/No) Reason:

(a) $h : \mathbb{N} \to \mathbb{N}$, where $h(n) = \sqrt{n}$.

No (Yes/No) Reason: $\sqrt{2}$ is not a natural number
It is irrational.

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14. Prove or disprove that the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 4n + 1$ is one-to-one.

By Contrapositive
Let $n_1, n_2 \in \mathbb{N}$.
Assume $f(n_1) = f(n_2)$.
then $4n_1 + 1 = 4n_2 + 1$
$4n_1 = 4n_2$
$n_1 = n_2$
and so f is one-to-one $\square$

-or-

By direct proof.
Let $n_1, n_2 \in \mathbb{N}$.
Assume $n_1 \neq n_2$
Then $4n_1 \neq 4n_2$
$4n_1 + 1 \neq 4n_2 + 1$
$f(n_1) \neq f(n_2)$
and so f is one-to-one
$\square$