PRINT Last name: Key First name: ______

Signature: ______ Student ID: ______

Math 230 Exam 1, Spring 2008

Show work for full credit. Partial credit for good proof structure even if proof is not correct.

7

1. A couple of weeks ago, I used the logical equivalence $p \to (q \vee r) \equiv (p \wedge \neg q) \to r$ to simplify a proof in a paper. Prove that this actually is a logical equivalence.

way 1

$p \to (q \vee r)$ is False exactly when p is true and $q \vee r$ is False, or p true, q false, r false.

$(p \wedge \neg q) \to r$ is False exactly when $p \wedge \neg q$ true, r false, or p true, q false, r false. Same

way 2

p	q	r	$q \vee r$	$p \to (q \vee r)$	$p \wedge \neg q$	$(p \wedge \neg q) \to r$
T	T	T	T	T	F	T
T	T	F	T	T	F	T
T	F	T	T	T	T	T
T	F	F	F	F	T	F
F	T	T	T	T	F	T
F	T	F	T	T	F	T
F	F	T	T	T	F	T
F	F	F	F	T	F	T

same column

7

2. Recall that a logical proposition involving only propositional variables, negations, and conjunctions (e.g., $p \wedge q \wedge \neg r$) is true for only one row of its truth table. Using this as a building block, find a proposition having the following truth table.

p	q	r	??
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	T

$(p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$

$\neg q \wedge (p \vee \neg r)$

6

3. Write the converse, inverse, and contrapositive of the following: "The team wins if the quarterback can run."

if (the quarterback can run) then (the team wins)

Contrapositive: If the team doesn't win, the quarterback can't run.

Converse: If the team wins then the quarterback can run.

Inverse: If the quarterback can't run then the team doesn't win.

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4. Circle the negation of the proposition $\forall x \left((P(x) \vee Q(x)) \to \neg R(x) \right)$.

negates to $\exists x$

$\neg(p \to q) \equiv p \wedge \neg q$

(a) $\forall x(\neg R(x) \to (P(x) \vee Q(x)))$
(b) $\exists x(\neg P(x) \vee \neg Q(x) \vee \neg R(x))$
(c) $\forall x((\neg P(x) \wedge \neg Q(x)) \vee \neg R(x))$
(d) $\exists x((P(x) \vee Q(x)) \wedge R(x))$
(e) $\forall x(R(x) \to (\neg P(x) \wedge \neg Q(x)))$

6

5. Write $\mathbb{T}$ next to the tautology. Write $\mathbb{F}$ next to the contradiction. Do nothing for the rest of the propositions.

 $\mathbb{T}$ (a) $p \vee (q \vee \neg p)$ p true → true   p false → true

 ____ (b) $p \wedge (r \vee p)$

 $\mathbb{F}$ (c) $(p \leftrightarrow q) \wedge (p \oplus q)$ since $\neg(p \Leftrightarrow q) \equiv p \oplus q$

 ____ (d) $q \rightarrow \neg q$

 ____ (e) $(p \oplus q) \wedge \neg p$

7

6. Suppose the variable x represents students, y represents courses, and $T(x, y)$ means "x is taking y". Next to the quantified statement, write the number of the equivalent English sentence.

2	$\forall y \exists x\, T(x, y)$	(1)	No student is taking any course.
3	$\exists x \forall y\, T(x, y)$	(2)	Every course is being taken by at least one student.
1	$\neg(\exists x \exists y\, T(x, y))$	(3)	Some student is taking every course.
4	$\neg(\forall x \neg(\forall y\, \neg T(x, y)))$	(4)	Some students are taking no courses.

7

7. Explain the case for truth values of p and q that shows that the following argument is not valid.

$$\begin{array}{l} p \rightarrow q \\ \underline{q \qquad} \\ \therefore p \end{array}$$

in the case q true and p false, we have both hypotheses true, but conclusion false.

$$\begin{array}{l} p \rightarrow q : F \rightarrow T \text{ is } T \\ \underline{q : T \qquad\quad} \\ \therefore p : F \end{array}$$

Invalid argument

7

8. Prove the following statement. If n^3 is an even integer then n is even. (Hint: is direct, contrapositive or contradiction easier?)

Proof (contrapositive)

Let n be an integer.
assume n is odd.
Then $n = 2k+1$ for some $k \in \mathbb{Z}$.

$$n^3 = (2k+1)^3 = (2k+1)^2(2k+1)$$
$$= (4k^2+4k+1)(2k+1)$$
$$= (8k^3+8k^2+2k+4k^2+4k)+1$$
$$= 2(4k^3+4k^2+k+2k^2+2k)+1$$

and therefore n^3 is odd by closure of addition, multiplication in $\mathbb{Z}$. $\square$

9

9. Prove that the equation $4x^2 + y^2 = 14$ has no integer solutions.

Proof by exhaustive cases. Since $0 \le 4x^2 \le 14$ forces $x = 0, \pm 1$ and $0 \le y^2 \le 14$ forces $y = 0, \pm 1, \pm 2, \pm 3$, and squaring removes the minus sign, we need only check 8 cases.

x \ y	0	1	2	3
0	0	1	4	9
1	4	5	8	13

therefore the equation has no integer solutions

9

10. Recall that by definition $A \oplus B = (A - B) \cup (B - A)$. Prove by set membership table that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.

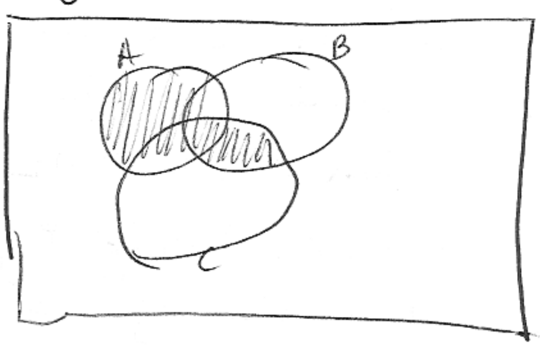
A	B	C	$B \oplus C$	$A \oplus (B \oplus C)$	$A \oplus B$	$(A \oplus B) \oplus C$
1	1	1	0	1	0	1
1	1	0	1	0	0	0
1	0	1	1	0	1	0
1	0	0	0	1	1	1
0	1	1	0	0	1	0
0	1	0	1	~~0~~ 1	1	1
0	0	1	1	~~0~~ 1	0	1
0	0	0	0	0	0	0

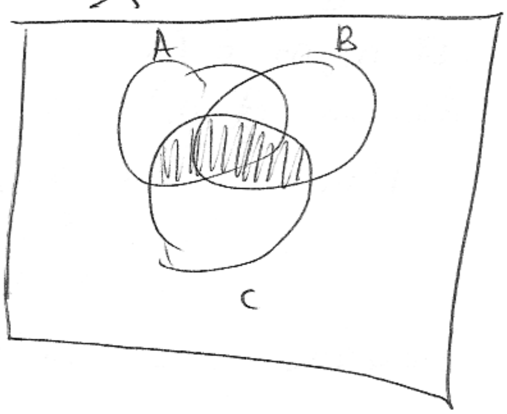
same

0

11. Draw two Venn diagrams (one for the left side, one for the right side) to justify which relationship, $\subseteq$, $=$, or $\supseteq$, is valid for the following pair of sets. Write the correct operator in the blank.

$A \cup (B \cap C)$ $\underline{\ \supseteq\ }$ $(A \cup B) \cap C$





10

12. In specifying a function, give the domain and codomain. Also give either the rule or clearly describe how the function is defined.

(a) Give an example of a function which is strictly decreasing.

(b) Give an example of a function which is (monotonic) decreasing but not strictly decreasing.

(a) $f: \mathbb{R} \to \mathbb{R}$
$f(x) = -x$

— or —

$f: \mathbb{R} \to \mathbb{R}$
$f(x) = e^{-x}$

many solutions

(b) $f: \mathbb{R} \to \mathbb{R}$
$f(x) = 1$

— or —

$f: \mathbb{R} \to \mathbb{R}$
$f(x) = \begin{cases} 0, & x \le 0 \\ -x, & x > 0 \end{cases}$

7

13. In the space provided, write whether or not the rule with the given domain and codomain describes a function, and if not give a brief reason why. Recall the natural numbers are $\mathbb{N} = \{0, 1, 2, \ldots\}$.

(a) $f : \mathbb{N} \to \mathbb{N}$, where $f(n) = \sqrt{n}$.

No (Yes/No) Reason: $\sqrt{2}$ is not ~~an~~ a natural number (it's irrational)

(b) $h : \mathbb{R} \to \mathbb{R}$, where $h(x) = \sqrt{x}$.

Yes (Yes/No) Reason:

(c) $g : \mathbb{N} \to \mathbb{N}$, where $g(n)$ equals any integer greater than n.

No (Yes/No) Reason: "any integer..." is not specific enough to define a single image of n.

7

14. Prove or disprove that the function $g : \mathbb{N} \to \mathbb{N}$ defined by $g(n) = 5n + 2$ is one-to-one.

By contrapositive.

Let $n_1, n_2 \in \mathbb{N}$.
Assume $g(n_1) = g(n_2)$.
then $5n_1 + 2 = 5n_2 + 2$
$5n_1 = 5n_2$
$n_1 = n_2$.
Therefore g is one-to-one □

— or —

By direct proof.

Let $n_1, n_2 \in \mathbb{N}$.
Assume $n_1 \neq n_2$.
Then $5n_1 \neq 5n_2$
and $\underbrace{5n_1 + 2}_{g(n_1)} \neq \underbrace{5n_2 + 2}_{g(n_2)}$.
Therefore g is one-to-one □