Math 230 Exam 1, Spring 2008

Show work for full credit. Partial credit for good proof structure even if proof is not correct.

1. A couple of weeks ago, I used the logical equivalence \( p \to (q \lor r) \equiv (p \land \neg q) \to r \) to simplify a proof in a paper. Prove that this actually is a logical equivalence.

   \( p \to (q \lor r) \) is false exactly when

   - \( p \) is true and \( q \lor r \) is false,
   - \( p \) false, \( q \) true, \( r \) false.

   \((p \land \neg q) \to r \) is false exactly when

   - \( p \) true, \( q \) false, \( r \) false.

   \((p \land \neg q) \lor r \) is false exactly when

   - \( p \) false, \( q \) true, \( r \) false.

   \((p \land \neg q) \lor r \) is false exactly when

   - \( p \) false, \( q \) false, \( r \) false.

   \((p \land \neg q) \lor r \) is false exactly when

   - \( p \) false, \( q \) false, \( r \) false.

2. Recall that a logical proposition involving only propositional variables, negations, and conjunctions (e.g., \( p \land q \land \neg r \)) is true for only one row of its truth table. Using this as a building block, find a proposition having the following truth table.

   \[
   \begin{array}{c|c|c|c|c|c|c|c|c}
   p & q & r & ?? & (p \land \neg q) & (p \land \neg q) & (p \land \neg q) & (p \land \neg q) \\
   \hline
   T & T & T & F & T & T & T & T \\
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   \end{array}
   \]

3. Write the converse, inverse, and contrapositive of the following: "The team wins if the quarterback can run."

   Contrapositive: If the team doesn't win, the quarterback can't run.
   Converse: If the team wins then the quarterback can run.
   Inverse: If the quarterback can't run then the team doesn't win.

4. Circle the negation of the proposition \( \forall x (P(x) \lor Q(x)) \to \neg R(x) \).

   (a) \( \forall x (\neg R(x) \to (P(x) \lor Q(x))) \)
   (b) \( \exists x (\neg P(x) \lor \neg Q(x) \lor \neg R(x)) \)
   (c) \( \forall x ((\neg P(x) \land \neg Q(x)) \lor \neg R(x)) \)
   (d) \( \exists x ((P(x) \lor Q(x)) \land R(x)) \)
   (e) \( \forall x (R(x) \to (\neg P(x) \land \neg Q(x))) \)
5. Write $T$ next to the tautology. Write $F$ next to the contradiction. Do nothing for the rest of the propositions.

(a) $p \lor (q \lor \neg p)$
(b) $p \land (r \lor p)$
(c) $(p \leftrightarrow q) \land (p \lor q)$ since $(p \leftrightarrow q) \equiv p \oplus q$
(d) $q \implies \neg q$
(e) $(p \lor q) \land \neg p$

6. Suppose the variable $x$ represents students, $y$ represents courses, and $T(x, y)$ means "$x$ is taking $y$". Next to the quantified statement, write the number of the equivalent English sentence.

(2) $\forall x \exists y T(x, y)$
(3) $\exists x \forall y T(x, y)$
(1) $\neg (\exists x \exists y T(x, y))$
(4) $\neg (\forall x \neg (\forall y \neg T(x, y)))$

7. Explain the case for truth values of $p$ and $q$ that shows that the following argument is not valid.

$p \implies q$
$q$
\[ \therefore p \]

In the case $q$ true and $p$ false, we have both hypotheses true, but $p \implies q$: $F \implies T$ is $T$ conclusion false.

$p \implies q$
$q$
\[ \therefore p \]

Invalid argument

8. Prove the following statement. If $n^3$ is an even integer then $n$ is even. (Hint: is direct, contrapositive or contradiction easier?)

Proof (contrapositive)

Let $n$ be an integer.
Assume $n$ is odd.
Then $n = 2k + 1$ for some $k \in \mathbb{Z}$.

\[ n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 \]

and therefore $n^3$ is odd by closure of addition, multiplication $n \in \mathbb{Z}$.

9. Prove that the equation $4x^2 + y^2 = 14$ has no integer solutions.

Proof by exhaustive cases. Since $0 \leq 4x^2 \leq 14$ forces $x = 0, \pm 1, \pm \sqrt{14}$ and $0 \leq y^2 \leq 14$ forces $y = 0, \pm 1, \pm 2, \pm 3$, and squaring removes the minus sign, we need only check 8 cases. 

\[
\begin{array}{c|c|c|c|c}
 x & 0 & 1 & 2 & 3 \\
 y & 0 & 1 & 2 & 3 \\
\hline
 0 & 0 & 1 & 4 & 9 \\
 1 & 4 & 5 & 8 & 13 \\
\end{array}
\]

Therefore, the equation has no integer solutions.

10. Recall that by definition $A \oplus B = (A - B) \cup (B - A)$. Prove by set membership table that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.

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11. Draw two Venn diagrams (one for the left side, one for the right side) to justify which relationship, $\subseteq$, $=$, or $\supseteq$, is valid for the following pair of sets. Write the correct operator in the blank.

$A \cup (B \cap C) \supseteq (A \cup B) \cap C$
12. In specifying a function, give the domain and codomain. Also give either the rule or clearly describe how the function is defined.

(a) Give an example of a function which is strictly decreasing.

(b) Give an example of a function which is (monotonic) decreasing but not strictly decreasing.

\[ f : \mathbb{R} \to \mathbb{R} \]
\[ f(x) = -x \]

\[ f : \mathbb{R} \to \mathbb{R} \]
\[ f(x) = 1 \]

\[ \ldots \]

\[ f : \mathbb{R} \to \mathbb{R} \]
\[ f(x) = e^{-x} \]

\[ \text{many solutions} \]

\[ f : \mathbb{R} \to \mathbb{R} \]
\[ f(x) = \begin{cases} 0, & x \leq 0 \\ -x, & x > 0 \end{cases} \]

13. In the space provided, write whether or not the rule with the given domain and codomain describes a function, and if not give a brief reason why. Recall the natural numbers are \( \mathbb{N} = \{0, 1, 2, \ldots\} \).

(a) \( f : \mathbb{N} \to \mathbb{N} \), where \( f(n) = \sqrt{n} \).

\[ \text{No (Yes/No)} \]
Reason: \( \sqrt{2} \) is not a natural number (it's irrational)

(b) \( h : \mathbb{R} \to \mathbb{R} \), where \( h(x) = \sqrt{x} \).

\[ \text{Yes (Yes/No)} \]
Reason:

(c) \( g : \mathbb{N} \to \mathbb{N} \), where \( g(n) \) equals any integer greater than \( n \).

\[ \text{No (Yes/No)} \]
Reason: "any integer..." is not specific enough to define a single image of \( n \).

14. Prove or disprove that the function \( g : \mathbb{N} \to \mathbb{N} \) defined by \( g(n) = 5n + 2 \) is one-to-one.

By contrapositive.

Let \( n_1, n_2 \in \mathbb{N} \).
Assume \( g(n_1) = g(n_2) \).
Then \( 5n_1 + 2 = 5n_2 + 2 \). 
\[ 5n_1 = 5n_2 \]
\[ n_1 = n_2. \]

Therefore \( g \) is one-to-one.

By direct proof.

Let \( n_1, n_2 \in \mathbb{N} \).
Assume \( n_1 \neq n_2 \).
Then \( 5n_1 \neq 5n_2 \) and \( 5n_1 + 2 \neq 5n_2 + 2 \).

Therefore \( g \) is one-to-one.