100 points. Show work on #4-14 for full credit!

1. (6 pts) For each set on the left, in the blank write the letter of the set on the right that is equal to it. You may assume that each set on the left matches exactly one set on the right.

c	$\{n \in \mathbb{Z} \mid 2 \text{ divides } n \text{ and } 6 \text{ divides } n\}$	**a.** $12\mathbb{Z}$
a	$3\mathbb{Z} \setminus \{12n + 3t \mid n \in \mathbb{Z} \text{ and } t \in \{1,2,3\}\}$	**b.** $\{n^2 \mid n \in \mathbb{Z}^+\}$
d	$\{n + m \mid n \text{ and } m \text{ are consecutive } \textbf{even} \text{ integers}\}$	**c.** $6\mathbb{Z}$
b	$\{\sum_{i=0}^{n}(2i+1) \mid n \in \mathbb{Z}_{\geq 0}\}$	**d.** $\{4k+2 \mid k \in \mathbb{Z}\}$

2. (8 pts) This question concerns counting the total number of possible functions of a certain type. For each type of function on the left, in the blank write the letter of the number on the right which gives the total number of functions of that type. Use the following definition:
Let $k, n \in \mathbb{Z}^+$ such that $n \geq k$. A function $f : \{1, \ldots, k\} \to \{1, \ldots, n\}$ is said to be *strictly increasing* if for all $x_1, x_2 \in \{1, \ldots, k\}$, if $x_1 < x_2$ then $f(x_1) < f(x_2)$.

c	The number of functions $f : \{1, \ldots, k\} \to \{1, \ldots, n\}$	**a.** $\binom{n}{k}$
e	The number of injective functions $f : \{1, \ldots, k\} \to \{1, \ldots, n\}$	**b.** $n^k - \frac{n!}{(n-k)!}$
a	The number of strictly increasing functions $f : \{1, \ldots, k\} \to \{1, \ldots, n\}$	**c.** n^k
b	The number of functions $f : \{1, \ldots, k\} \to \{1, \ldots, n\}$ which are not injective	**d.** $n^k - \binom{n}{k}$
d	The number of functions $f : \{1, \ldots, k\} \to \{1, \ldots, n\}$ which are not strictly increasing	**e.** $\frac{n!}{(n-k)!}$

3. (6 pts) Consider the statement "If a divides b and b divides a, then $a = b$ or $a = -b$."
(a) (2 pts) Write the contrapositive of the statement.
(b) (2 pts) Write the converse of the statement.
(c) (2 pts) Write the negation of the statement.

(a) If $a \neq b$ and $a \neq -b$ then $a \nmid b$ or $b \nmid a$.
(b) If $a = b$ or $a = -b$ then $a|b$ and $b|a$.
(c) $a|b$ and $b|a$ and $a \neq b$ and $a \neq -b$.

4. (6 pts) Let A, B, and C be statements.
(a) (4 pts) Prove that $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$ is a tautology.
(b) (2 pts) Briefly describe how this tautology could be used in a proof of the statement $A \Rightarrow C$.

A	B	C	A⇒B	B⇒C	(A ⇒ B) ∧ (B ⇒ C)	A⇒C	statement
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

(b) This tautology shows A proof of A⇒C can be achieved by finding a statement B such that A⇒B and B⇒C can be proved.

5. (6 pts) Define the binary operation $*$ on $\mathbb{R}$ by $a * b = a + b/2$. Settle the following assertions with a proof or disproof/counterexample.

(a) (3 pts) $*$ is associative.

(b) (3 pts) There is an identity element in $\mathbb{R}$ for $*$.

(a) (*Counter example*) Pick $a = 0$, $b = 1$, and $c = 2$. Note that $a, b, c \in \mathbb{R}$.
$0 * (1 * 2) = 0 * (1 + \frac{1}{2}) = 0 + \frac{3/2}{2} = \frac{3}{4}$.
$(0 * 1) * 2 = (0 + \frac{1}{2}) * 2 = \frac{1}{2} + \frac{2}{2} = \frac{3}{2}$.
Therefore $0 * (1 * 2) \neq (0 * 1) * 2$ and $*$ is not associative.

(b) (*Disproof*) Suppose to the contrary that there is an identity element e in $\mathbb{R}$ for $*$.
$1 \in \mathbb{R}$, so $e * 1 = 1 \Rightarrow e + \frac{1}{2} = 1 \Rightarrow e = \frac{1}{2}$.
Also, $1 * e = 1$, so $1 + \frac{e}{2} = 1 \Rightarrow e = 0$.
Having both $e = \frac{1}{2}$ and $e = 0$ is a contradiction.
Therefore $*$ has no identity in $\mathbb{R}$.

6. (6 pts) Define the function $f : \mathbb{R}^* \to \mathbb{R}^*$ by $f(x) = 1/x$. Give an example of a function $h : \mathbb{R} \to \mathbb{R}^*$ such that h is injective and $f \circ h$ is not surjective.

Choosing something like $h(x) = e^x$ or $h(x) = \arctan x + \frac{\pi}{2}$ makes $h(x)$ injective and is defined on all of $\mathbb{R}$. Furthermore, these functions do not map anything to 0 as required by the codomain being $\mathbb{R}^*$, and do not have, say, negative numbers in their image, so that $f \circ h$ is not surjective.

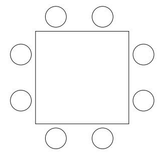
7. (6 pts) Compute the coefficient of x^4 in $(2x + 3)^{10}$.

By the Binomial Theorem, for all $a, b \in \mathbb{R}$ and for all $n \in \mathbb{Z}_{\geq 0}$, $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$. Therefore $(2x + 3)^{10} = \sum_{i=0}^{10} \binom{10}{i} (2x)^i 3^{10-i}$. The x^4 term corresponds to $i = 4$, and so the coefficient of x^4 is $\binom{10}{4} 2^4 3^6$.

8. (6 pts) Determine the number of relations R on the set $\{a, b, c, d\}$ such that R is symmetric and a is not related to b.

There are 4 elements for which to either choose or not choose to be related to themselves: aRa, bRb, cRc, and dRd. This is 2^4 choices total. Because the relation we are building must be symmetric, we either choose both xRy and yRx simultaneously, or neither. Because a is not related to b, neither is b related to a. Of the remaining 5 pairs $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$, we can either choose both in the pair to be related to each other or neither, in 2^5 ways. These two types of relationships are independent, so the total number of relations is $2^4 \cdot 2^5 = 2^9$.

9. (6 pts) A perfectly square table has 2 seats on each side, so that the table looks exactly the same when rotated 90^o, 180^o, 270^o, or 360^o. How many ways are there to seat 4 boys and 4 girls at the table so that no two boys sit next to each other? (Two seatings are are equal if one can be rotated into the other by rotations in increments of 90^o. A seated person is "next to" both the person to the left and to the right.)

Refer to the diagram for this problem: 

Place the first boy at the top of the table for free. Now choose whether he is to sit in the left or right spot (2 ways). Going around the table clockwise, choose from 4 girls, then from 3 remaining boys, then from 3 girls, etc. The total number of ways is $2 \cdot 4! \cdot 3!$.

10. (8 pts) Prove that for all $a, b \in \mathbb{Z}$, if there exist integers x and y such that $ax + by = 1$, then $(a, b) = 1$. (Recall (a, b) is the gcd of a and b.)

(*Proof*) Let $a, b \in \mathbb{Z}$ and suppose there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$.
Define $d = (a, b)$. By definition of gcd, $d|a$ and $d|b$.
Since $d|a$, then $d|ax$. Since $d|b$, then $d|by$.
Since $d|ax$ and $d|by$, $d|(ax + dy)$, and so $d|1$.
The only divisors of 1 are 1 and -1.
The gcd is defined to be a positive integer, so $d = 1$.
Therefore $(a, b) = 1$ $\square$

11. (8 pts) Let A and B be finite nonempty sets, and let $f : A \to B$ be a function. Prove that if $|A| < |B|$ then f is not surjective.

(*Proof*) Suppose $|A| < |B|$ and assume to the contrary that f is surjective.
Note that $B = \cup_{a \in A} f(a)$.
Setting $n = |B|$ and $k = |A|$, we have that B is a set of size n which is the union of k sets, where $n > k$.
By the Pigeonhole Principle, some set in the union has size at least 2.
Therefore there exists an $a \in A$ such that $|f(\{a\})| \geq 2$.
This contradicts the definition of a function, which states that one domain element can map to only one codomain element.
Therefore f is not surjective. $\square$

(This is only one proof, and others are possible.)

12. (8 pts) Find a set A such that $A = \bigcup_{i=1}^{\infty} [0, i)$, and prove that they are equal. (The set A must be given in interval notation. Any variation of using $\bigcup_{i=1}^{\infty} [0, i)$ as A and proving $\bigcup_{i=1}^{\infty} [0, i) = \bigcup_{i=1}^{\infty} [0, i)$ will receive no credit.)

Choose $A = [0, \infty)$.
Proof of $[0, \infty) = \bigcup_{i=1}^{\infty} [0, i)$:
$(\subseteq)$ Let $x \in [0, \infty)$.
By the Archimedean Principle, there exist an integer n such that $x < n$.
Since $0 \leq x$, $x \in [0, n)$.

Since $n \in \mathbb{Z}^+$, $x \in \bigcup_{i=1}^{\infty}[0,i)$.
($\supseteq$) Let $x \in \bigcup_{i=1}^{\infty}[0,i)$.
Then there exists an $i \in \mathbb{Z}^+$ such that $x \in [0,i)$.
Therefore $0 \leq x < i$.
Since $x < i$ and $i < \infty$, then $x < \infty$.
Therefore $x \in [0,\infty)$. $\square$

13. (10 pts) Let A and B be nonempty sets, and let $f : A \to B$ be a bijection. Suppose $\mathcal{P}$ is a partition of B.

(a) (2 pts) Let $Y \subseteq B$. Give the set definition of the inverse image $f^{-1}(Y)$ of Y.

(b) (7 pts) Prove that the sets in the collection $\{f^{-1}(Y) : Y \in \mathcal{P}\}$ are pairwise disjoint.

(c) (1 pt) Is the collection $\{f^{-1}(Y) : Y \in \mathcal{P}\}$ a partition of A (Yes/No)?

(a) $f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$.

(b) (*Proof*) Let $S, T \in \{f^{-1}(Y) : Y \in \mathcal{P}\}$ and suppose $S \cap T \neq \emptyset$.
There must be some $Y_1 \in \mathcal{P}$ such that $S = f^{-1}(Y_1)$.
There must be some $Y_2 \in \mathcal{P}$ such that $T = f^{-1}(Y_2)$.
Since $S \cap T \neq \emptyset$, there is some $a \in S \cap T$.
Therefore $a \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$, and so $f(a) \in Y_1$ and $f(a) \in Y_2$.
But Y_1 and Y_2 are elements of a partition, so $Y_1 = Y_2$.
Therefore $S = T$.
$S \cap T \neq \emptyset \Rightarrow S = T$, and so the elements of $\{f^{-1}(Y) : Y \in \mathcal{P}\}$ are pairwise disjoint. $\square$

(c) Yes. This is because a bijection on a finite set A can be thought of as simply renaming the elements of a. A collection that is a partition after renaming is also a partition before renaming. You can prove the other two properties of being a partition, i.e., no $f^{-1}(Y)$ is empty (because f is surjective), and the union of all the $f^{-1}(Y)$'s is A (because f is a function and maps every $a \in A$ to some $b \in B$).

14. (10 pts) Assume $f(1) = 3$ and $f(n+1) = 2f(n) + 2^{2n}$ for all $n \in \mathbb{Z}^+$. Prove $f(n) = 2^{n-1} + 2^{2n-1}$ for all $n \in \mathbb{Z}^+$.

(*Proof*) For all $n \in \mathbb{Z}^+$, let $P(n)$ be the statement $f(n) = 2^{n-1} + 2^{2n-1}$.
$2^{1-1} + 2^{2\cdot 1-1} = 2^0 + 2^1 = 3$, and so $P(1)$ is true.
Let $k \in \mathbb{Z}^+$ and assume $P(k)$ is true.
$f(k+1) = 2f(k) + 2^{2k} = 2(2^{k-1} + 2^{2k-1}) + 2^{2k} = 2^k + 2^{2k} + 2^{2k} = 2^{(k+1)-1} + 2^{2(k+1)-1}$.
Therefore $P(k+1)$ is true.
Therefore by the first principle of mathematical induction, for all $n \in \mathbb{Z}^+$, $P(n)$ is true. $\square$