SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. **(16pts)** For each test and each series below, check exactly one box of the four possible choices. Recall that not all choices are valid for all tests. A series would fail the preconditions for the alternating series test, for example, if the the signs are not alternating. (No partial credit. You do not have to show work.)

	(i) Divergence Test	(ii) Integral Test	(iii) Geometric Series Test	(iv) Alternating Series Test
a) $\sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{7^n}$	☐ fails pre-conditions ☐ diverges ☒ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☐ fails pre-conditions ☐ diverges ☐ inconclusive ☒ converges	☐ fails pre-conditions ☐ diverges ☐ inconclusive ☒ converges
(b) $\sum_{n=1}^{\infty} \frac{1}{n}$	☐ fails pre-conditions ☐ diverges ☒ inconclusive ☐ converges	☐ fails pre-conditions ☒ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges
(c) $\sum_{n=1}^{\infty} \frac{5 \cdot 4^{n-1}}{3^n}$	☐ fails pre-conditions ☒ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☐ fails pre-conditions ☒ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges
(d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$	☐ fails pre-conditions ☐ diverges ☒ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☐ fails pre-conditions ☐ diverges ☐ inconclusive ☒ converges

(a)(i) $\lim_{n\to\infty} \frac{(-1)^n 5^n}{7^n} = 0$. (ii) $f(x) = \frac{(-1)^x 5^x}{7^x}$ not always positive. (iii) $a = -\frac{5}{7}$, $r = -\frac{5}{7}$, $|r| < 1$

(iv) signs alternate, $\left|\frac{(-1)^{n+1} 5^{n+1}}{7^{n+1}}\right| \le \left|\frac{(-1)^n 5^n}{7^n}\right|$, and $\lim_{n\to\infty} \frac{(-1)^5 5^n}{7^n} = 0$.

(b)(i) $\lim_{n\to\infty} \frac{1}{n} = 0$. (ii) $\int_1^\infty \frac{1}{x}\,dx = \infty$ (iii) ratio $\frac{a_{n+1}}{a_n} = \frac{1/n+1}{1/n} = \frac{n}{n+1}$ not constant

(iv) signs don't alternate

(c)(i) $\lim_{n\to\infty} \frac{5 \cdot 4^{n-1}}{3^n} = \infty$ (ii) $f(x) = 5 \cdot \frac{4^{x-1}}{3^x}$ not decreasing (iii) $a = \frac{5}{3}$ $r = \frac{4}{3} > 1$.

(iv) signs don't alternate

(d)(i) $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0$. (ii) $f(x) = \frac{(-1)^{x+1}}{x}$ not always positive (iii) ratio $\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2}/n+1}{(-1)^{n+1}/n} = \frac{-n}{n+1}$ not constant

(iv) signs alternate, $\left|\frac{(-1)^{n+2}}{n+1}\right| \le \left|\frac{(-1)^{n+1}}{n}\right|$, and $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0$.

2. **(10pts)** Find the center, radius and the **interval** of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^n\frac{(x+2)^n}{n\cdot 7^n}.$$

Ratio Test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(x+2)^{n+1}}{(n+1)\,7^{n+1}}}{\frac{(-1)^n(x+2)^n}{n\,7^n}}\right| = \lim_{n\to\infty}\left|\frac{(x+2)\,n}{(n+1)\,7}\right|$

$= \left|\frac{x+2}{7}\right|.$ $\left|\frac{x+2}{7}\right|<1$

$|x+2|<7$

$-7<x+2<7$

$-9<x<5$

converges on $-9<x<5$.

Endpoints $x=-9$: $\sum_{n=1}^{\infty}\frac{(-1)^n(-9+2)^n}{n\cdot 7^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(-7)^n}{n\cdot 7^n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n}$

$= \sum_{n=1}^{\infty}\frac{1}{n}$ diverges as p-series, $p=1$.

$x=5$: $\sum_{n=1}^{\infty}\frac{(-1)^n(5+2)^n}{n\cdot 7^n} = \sum_{n=1}^{\infty}\frac{(-1)^n 7^n}{n\cdot 7^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges by alt. series test

center $= -2$. radius $= 7$ interval: $(-9,5]$

3. **(10pts)** Use an appropriate test to determine the convergence of the series $\sum_{n=3}^{\infty}\frac{\ln n}{n}$. The preconditions of the test must be mentioned individually but do not have to be checked.

Integral test $f(x)=\frac{\ln x}{x}$ is positive, decreasing, continuous for $x\geq 3$

$\int_3^{\infty}\frac{\ln x}{x}\,dx = \int_{x=3}^{\infty} u\,du = \frac{u^2}{2}\Big|_{x=3}^{\infty} = \frac{(\ln x)^2}{2}\Big|_3^{\infty} = \infty$

$u=\ln x$

$du=\frac{dx}{x}$

therefore $\sum_{n=3}^{\infty}\frac{\ln n}{n}$ diverges by integral test.

alternatively, Comparison test: comparison to $\sum_{n=3}^{\infty}\frac{1}{n}$. $0<1<\ln n$ for $n\geq 3$, so $0<\frac{1}{n}<\frac{\ln n}{n}$ for $n\geq 3$.

$\sum_{n=3}^{\infty}\frac{1}{n}$ diverges as p-series, $p=1$,

so $\sum_{n=3}^{\infty}\frac{\ln n}{n}$ diverges by comparison.

4. **(8pts)** Assume that $\sum_{n=1}^{\infty} a_n$ is a convergent series, with nth partial sum $s_n = 8 - \dfrac{3n}{4n+2}$. What is the sum of the series?

By definition, sum $s = \lim_{n\to\infty}\left(8 - \frac{3n}{4n+2}\right) =$ (4)

$\lim_{n\to\infty} 8 - \frac{3}{4+2/n} = 8 - \frac{3}{4} = \boxed{29/4}$ (4)

5. **(8pts)** Determine whether the sequence $\{\ln(3n) - \ln(6n+3)\}_{n=1}^{\infty}$ converges or diverges. Either compute the limit or the nature in which the sequence diverges.

$\lim_{n\to\infty}\left(\ln(3n) - \ln(6n+3)\right)$ (3) $= \lim_{n\to\infty} \ln\left(\frac{3n}{6n+3}\right)$ (2) $= \ln\left(\lim_{n\to\infty} \frac{3n}{6n+3}\right)$ (2)

$= \ln\left(\lim_{n\to\infty} \frac{3}{6+3/n}\right) = \ln\left(\frac{3}{6}\right) = \boxed{\ln\frac{1}{2}}$ (1)

6. **(8pts)** Determine the sum of the series $\sum_{n=0}^{\infty} \dfrac{(-1)^n 4^n}{7^{n+1}}$, or show that it diverges.

a geometric series. (2) first term $= \frac{(-1)^0 4^0}{7^{0+1}} = \frac{1}{7}$ (2)

second term $= \frac{(-1)^1 4^1}{7^{1+1}} = \frac{-4}{7^2}$ (1)

$a = \frac{1}{7}$

$r = \frac{-4/7^2}{1/7} = \frac{-4}{7}$ (1). $|r|<1$, so converges to

(2)

$\frac{a}{1-r} = \frac{\frac{1}{7}}{1-\left(-\frac{4}{7}\right)} = \frac{1/7}{11/7} = \boxed{\frac{1}{11}}$

7. **(10pts)** For the two parts of this question, a complete technical proof is not necessary, but your answers must be justified.

(a) For what values of r does the sequence $\left\{\dfrac{n^r}{\sqrt{n}}\right\}_{n=1}^{\infty}$ converge?

(b) For what values of t does the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^t}$ converge?

(a) $\left\{\frac{n^r}{\sqrt{n}}\right\} = \left\{\frac{n^r}{n^{1/2}}\right\} = \left\{n^{r-\frac{1}{2}}\right\}$ converges when $r-\frac{1}{2} \le 0$, or $\boxed{r \le \frac{1}{2}}$

(b) $\sum \frac{\sqrt{n}}{n^t} = \sum \frac{n^{\frac{1}{2}}}{n^t} = \sum \frac{1}{n^{t-\frac{1}{2}}}$ Converges when $t-\frac{1}{2} > 1$ or $\boxed{t > \frac{3}{2}}$

8. **(10pts)** Find an estimate for the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ which is within a tolerance of 10^{-4} of the actual sum. You do not have to completely simplify the estimate, but it must be totally clear what it is. (Hint: estimate, error, error bound, tolerance.)

estimate	error	error bound	tolerance
S_n	$R_n = S - S_n$	$R_n \le \int_n^\infty \frac{1}{x^2}dx$	10^{-4}

$$\int_n^\infty \frac{1}{x^2}dx = -\frac{1}{x}\Big|_n^\infty = \frac{1}{n}$$

Set RHS of error bound $\le$ tolerance

$$\frac{1}{n} \le 10^{-4}$$

$$n \ge 10^4$$

Thus $\boxed{S_{10000}}$ is guaranteed to be within 10^{-4} of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$.

9. **(10pts)** Determine whether the following series converge or diverge. You do not have to check the preconditions of the test you use, so long as those preconditions hold. Where appropriate, state how you are setting up the test, and the specific conclusion.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^n+5^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n n^2}$

(a) limit comparison to $\sum_{n=1}^{\infty} \frac{3^n}{5^n}$. 2

$$\lim_{n\to\infty} \frac{\frac{3^n}{4^n+5^n}}{\frac{3^n}{5^n}} = \lim_{n\to\infty} \frac{5^n}{4^n+5^n} = \lim_{n\to\infty} \frac{5^n/5^n}{4^n/5^n + 5^n/5^n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\frac{4}{5}\right)^n + 1} = 1.$$ 2

$\sum_{n=1}^{\infty} \frac{3^n}{5^n}$ converges; thus $\sum_{n=1}^{\infty} \frac{3^n}{4^n+5^n}$ also converges by limit comparison. 1

(b) Ratio Test. 2

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{3^{n+1}(n+1)^2}}{\frac{n!}{3^n n^2}} \right| = \lim_{n\to\infty} \frac{(n+1)\,n^2}{3(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{n^3+n^2}{3(n^2+2n+1)} = \lim_{n\to\infty} \frac{1+1/n}{3(1/n+2/n^2+1/n^3)} = \infty > 1.$$ 2

Therefore $\sum_{n=1}^{\infty} \frac{n!}{3^n n^2}$ diverges by the Ratio Test. 1

10. **(a) (5pts)** Give an example of a series which converges but does not converge absolutely.

(b) (5pts) Give an example of a series $\displaystyle\sum_{n=1}^{\infty} a_n$ such that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n!}$ converges but does not converge absolutely.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (Converges by Alt series test, but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges as p-series, $p=1$)

(b) set $\frac{a_n}{n!} = \frac{(-1)^n}{n}$

or $\boxed{a_n = \frac{(-1)^n n!}{n}}$. $\left(\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n n!}{n}\right)$

Then $\sum_{n=1}^{\infty} \frac{a_n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is as desired.