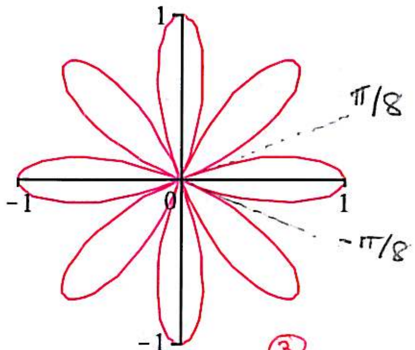
SHOW WORK FOR FULL CREDIT NO CALCULATORS

14

1. Find the area enclosed by one loop of the polar curve $r(\theta) = \cos(4\theta)$.



$\cos 4\theta = 0$

$4\theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ 2

$\theta = -\frac{\pi}{8}, \frac{\pi}{8}, \ldots$

$\text{Area} = \int_{-\pi/8}^{\pi/8} \frac{\cos^2 4\theta}{2}\, d\theta = 2\int_0^{\frac{\pi}{8}} \frac{\cos^2 4\theta}{2}\, d\theta$ 1 2 1 1

$= \int_0^{\frac{\pi}{8}} \frac{1+\cos 8\theta}{2}\, d\theta = \frac{\theta}{2} + \frac{\sin 8\theta}{16}\Big|_0^{\pi/8} = \left(\frac{\pi}{16} + 0\right) - (0+0)$ 3 3

$= \boxed{\frac{\pi}{16}}$ 1

15

2. For this problem, use the differential equation for Newton's Law of Cooling:

$$\frac{dT}{dt} = k(T - T_s),$$

where T is the temperature, t is time, T_s is the ambient temperature, and k is a constant.

Question: Water at temperature 49°F is placed in a freezer which is at 15°F. After 1 hour, the water is at temperature 45°F. How long does it take the water to freeze, that is, to reach 32°F? (Partial credit for writing the general solution without solving for it.)

$\frac{dT}{(T-T_s)} = k\,dt$

4 $\ln|T-T_s| = kt + C$

$|T-T_s| = e^{kt+C}$

$T - T_s = Ae^{kt}$

2 $\boxed{T = T_s + Ae^{kt}}$ gen'l Soln.

Info 1 $49 = 15 + Ae^{k\cdot 0}$

$T(0)=49$ + $T_s = 15$ $34 = A$

$\boxed{T = 15 + 34e^{kt}}$ 2

Info 2 $T(1)=45$ $45 = 15 + 34e^{k\cdot 1}$

$30 = 34e^k$

$\frac{15}{17} = e^k$

$\boxed{\ln\frac{15}{17} = k}$ 3

particular solution	$T = 15 + 34e^{(\ln\frac{15}{17})t}$

Question 3 solve $T(t) = 32$ for t

$32 = 15 + 34e^{\ln(\frac{15}{17})t}$

$17 = 34e^{(\ln 15/17)t}$

$\frac{1}{2} = e^{(\ln\frac{15}{17})t}$

$\ln\frac{1}{2} = \left(\ln\frac{15}{17}\right)t$

3 $t = \boxed{\frac{\ln\frac{1}{2}}{\ln\frac{15}{17}}}$

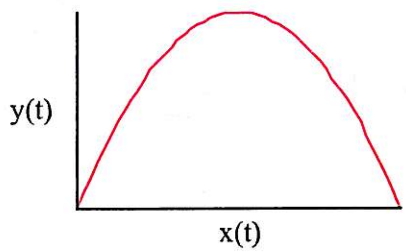
$(\doteq 5.54 \text{ hrs})$

15

3. A pumpkin is fired so that its horizontal and vertical position at time t in seconds is given by $x(t)$ and $y(t)$, respectively, where

$$\begin{aligned} x(t) &= \frac{13}{2}t \\ y(t) &= \frac{36}{5}t - \frac{1}{5}t^2 \\ t &\geq 0. \end{aligned}$$

3 (a) Find the maximum height (vertical position) of the projectile.
4 (b) Find the time at which this height is achieved.
4 (c) If the ground is at height 0, determine the distance the pumpkin is launched.
4 (d) Is the angle of launch with respect to horizontal greater or less than 45°?



(a) $\frac{dy}{dt} = \frac{36}{5} - \frac{2}{5}t$

$0 = \frac{36}{5} - \frac{2}{5}t$

$36 = 2t$

$\boxed{18 = t}$ (b)

$y(18) = \boxed{\frac{36 \cdot 18}{5} - \frac{1}{5}18^2}$ (a)

$\left(= \frac{324}{5}\right)$

(c) $y(t) = 0 = \frac{36}{5}t - \frac{1}{5}t^2$

$0 = (36 - t)\,t$

$t = 0, \underline{36}$

$x(36) = \frac{13}{2} \cdot 36 = 13 \cdot 18$

$= \boxed{234}$

(d) in other words, compare $\left.\frac{dy}{dx}\right|_{t=0}$ to 1.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{36}{5} - \frac{2}{5}t}{\frac{13}{2}}$

@ $t=0$: $\frac{\frac{36}{5}}{\frac{13}{2}} = \frac{36 \cdot 2}{5 \cdot 13} = \frac{72}{65} > 1$.

$\boxed{\text{greater than } 45°}$

14

4. Find the general solution to the differential equation

$$x^3y' + 4x^2y = e^x\,,$$

where $x > 0$ (this means assume $x > 0$ when deriving the general solution).

standard linear form: $y' + \frac{4}{x}y = \frac{e^x}{x^3}$

integrating factor: $e^{\int \frac{4}{x}dx} = e^{4\ln|x|}$

$= e^{4\ln x}$ since $x > 0$

$= (e^{\ln x})^4 = x^4$

$y'x^4 + 4x^3y = xe^x$

$(yx^4)' = xe^x$

$\int (yx^4)'\,dx = \int xe^x\,dx$

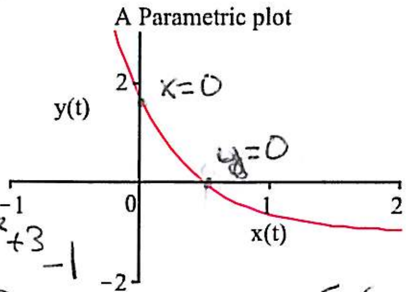
$yx^4 = xe^x - e^x + C$

$\boxed{y = \frac{e^x}{x^3} - \frac{e^x}{x^4} + \frac{C}{x^4}}$

x	e^x
1	e^x
	e^x

14 5. Find the area in the first quadrant bounded between the origin and the parametric curve given by

$$\begin{aligned} x(t) &= t^2 - 1, &= f(t) \\ y(t) &= e^{-2t^2+3} - 1, &= g(t) \\ t &\geq 0. \end{aligned} \quad (1)$$



$x(t) = 0 = t^2 - 1$

$1 = t^2$

$\pm 1 = t$

$\boxed{t = 1}$ (2)

$y(t) = 0 = e^{-2t^2+3} - 1$

$1 = e^{-2t^2+3}$

$0 = -2t^2 + 3$

$\frac{3}{2} = t^2$

$\pm\sqrt{\frac{3}{2}} = t$

$\boxed{t = \sqrt{\frac{3}{2}}}$ (2)

(4) $\text{Area} = \int_a^b g(t) f'(t)\,dt$

$f'(t) = 2t$ (1)

$\text{Area} = \int_1^{\sqrt{3/2}} \left(e^{-2t^2+3} - 1\right) 2t\,dt$ (1) $= 2\int_1^{\sqrt{3/2}} \left(e^{-2t^2+3} - 1\right) t\,dt$

$u = -2t^2 + 3$

$du = -4t\,dt$

$-\frac{du}{2} = 2t\,dt$

$= \int_{t=1}^{t=\sqrt{3/2}} -\frac{(e^u - 1)\,du}{2} = -\frac{e^u}{2} + \frac{u}{2}\Big|_{t=1}^{t=\sqrt{3/2}}$

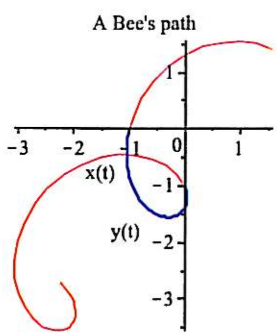
$= -\frac{\exp(-2t^2+3)}{2} + \frac{(-2t^2+3)}{2}\Big|_1^{\sqrt{3/2}}$

$= \left(-\frac{1}{2} + 0\right) - \left(-\frac{\exp(1)}{2} + \frac{1}{2}\right) = \boxed{\frac{e}{2} - 1}$ (4)

14 6. A bee buzzes along a path with parametric description

$$\begin{aligned} x(t) &= \frac{t}{2} - \cos\left(\frac{\pi t}{2}\right), \\ y(t) &= \frac{t}{2} + \sin\left(\frac{\pi t}{2}\right). \end{aligned}$$

Set up (**but do not integrate**) an integral representing the arc length of the portion of the path starting at $(x_1, y_1) = (0, -1)$ and ending at $(x_2, y_2) = (-1, 0)$. The integrand should be a function of t which contains no derivatives.



$\text{Arc length} =$ (6) $\int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\,dt$

$\frac{dy}{dt} = \frac{1}{2} + \frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)$ (1) $\quad \frac{dx}{dt} = \frac{1}{2} + \frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right)$ (1)

$(x_1, y_1) = (0, 1)$

$x(t) = 0 = \frac{t}{2} - \cos\left(\frac{\pi t}{2}\right)$

$\frac{t}{2} = \cos\left(\frac{\pi t}{2}\right)$

(2) $\boxed{t = 2}$: $\frac{-2}{2} = \cos(-\pi)$

$-1 = -1$ ✓

$y(-2) = -1 + \sin(-\pi) = -1.$

$(x_2, y_2) = (-1, 0)$

$y(t) = 0 = \frac{t}{2} + \sin\left(\frac{\pi t}{2}\right)$

(2) $\boxed{t = 0}$: $0 = \frac{0}{2} + \sin(0)$ ✓

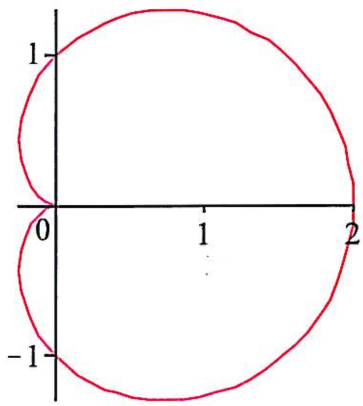
$x(0) = 0 - \cos(0) = -1.$

$\text{Arc length} =$ (2)

$$\boxed{\int_{-2}^{0} \sqrt{\left(\frac{1}{2} + \frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)\right)^2 + \left(\frac{1}{2} + \frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right)\right)^2}\,dt}$$

14 7. A cardiod is given in polar form by the function $r(\theta) = 1 + \cos(\theta)$ for $0 \leq \theta \leq 2\pi$. Find the **horizontal** tangents to the plot of the cardiod, carefully describing the behavior at the cusp (at the pole). (Hint: It may be helpful to convert expressions to either all cosines or all sines.)

Plot of a cardiod



$x(\theta) = (1+\cos\theta)\cos\theta = \cos\theta + \cos^2\theta$

$y(\theta) = (1+\cos\theta)\sin\theta = \sin\theta + \sin\theta\cos\theta$

$$\frac{dy}{d\theta} = \cos\theta + \cos^2\theta - \sin^2\theta$$
$$= \cos\theta + \cos^2\theta - (1-\cos^2\theta)$$
$$= \cos\theta + 2\cos^2\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1)$$

$\frac{dy}{d\theta} = 0$ when $2\cos\theta - 1 = 0$ or $\cos\theta + 1 = 0$

$\cos\theta = \frac{1}{2}$ $\quad\cos\theta = -1$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ $\quad\theta = \pi$

$$\frac{dx}{d\theta} = -\sin\theta + 2\cos\theta(-\sin\theta)$$
$$= -\sin\theta(1 + 2\cos\theta)$$

$\frac{dx}{d\theta} = 0$ when $\sin\theta = 0$ or $1 + 2\cos\theta = 0$

$\theta = \pi, 0, 2\pi$ $\quad\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

Therefore at $\theta = \boxed{\frac{\pi}{3}, \frac{5\pi}{3},}$ clear horizontal tangents.

10

At $\theta = \pi$, $\frac{dy}{dx}$ has form $\frac{0}{0}$. use L'Hospital's rule.

$$\lim_{\theta \to \pi^-} \frac{dy}{dx} \overset{L'H}{=} \lim_{\theta \to \pi^-} \frac{-\sin\theta - 4\cos\theta\sin\theta}{-\cos\theta(1+2\cos\theta) - \sin\theta(-2\sin\theta)} = \frac{0}{+1(-1)} = 0.$$

Therefore at ~~π~~, $\theta \to \pi^-$, a horizontal tangent in the limit.

4

same for $\theta \to \pi^+$, since $r(\theta) = r(-\theta)$

(function has horiz axis symmetry)