SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. Find the general solution to the differential equation

$$xy' + 2y = e^x\,,$$

where $x > 0$ (this means assume $x > 0$ when deriving the general solution).

standard linear form : $y' + \frac{2y}{x} = \frac{e^x}{x}$ (4)

integrating factor $e^{\int \frac{2}{x}dx} = e^{2\ln|x|}$ (4)

$= e^{2\ln x}$ since $x > 0$

$= \left(e^{(\ln x)}\right)^2 = x^2$

$y'x^2 + 2xy = xe^x$

$(yx^2)' = xe^x$

$\int (yx^2)'dx = \int xe^x\,dx$ (6)

$yx^2 = xe^x - e^x + C$

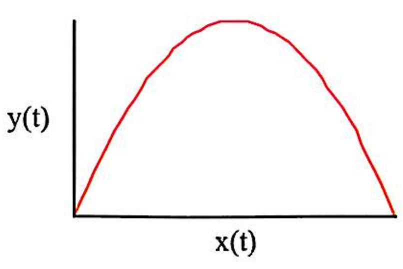
$\boxed{y = \frac{e^x}{x} - \frac{e^x}{x^2} + \frac{C}{x^2}}$

$\int$ | x | e^x
| 1 | e^x
| | e^x

(15) 2. A pumpkin is fired so that its horizontal and vertical position at time t in seconds is given by $x(t)$ and $y(t)$, respectively, where

$$\begin{aligned} x(t) &= \frac{117}{22}t \\ y(t) &= \frac{44}{10}t - \frac{1}{10}t^2 \\ t &\geq 0. \end{aligned}$$

(3) (a) Find the maximum height (vertical position) of the projectile.
(4) (b) Find the time at which this height is achieved.
(4) (c) If the ground is at height 0, determine the distance the pumpkin is launched.
(4) (d) Is the angle of launch with respect to horizontal greater or less than 45°?



(c) $y(t) = 0 = \frac{44}{10}t - \frac{1}{10}t^2$

$0 = t(44 - t)$

$t = 0, 44$

$x(44) = \frac{117}{22}\cdot 44 = \boxed{234}$

$y'(t) = \frac{44}{10} - \frac{2}{10}t$

$0 = \frac{44}{10} - \frac{2}{10}t$

$44 = 2t$

$\boxed{22 = t}$ (b)

(a)

$y(22) = \boxed{\frac{44}{10}\cdot 22 - \frac{1}{10}22^2 = \frac{242}{5}}$

(d) it's asking if $\left.\frac{dy}{dx}\right|_{t=0}$ is >1 or <1.

$\left.\frac{dy}{dx}\right|_{t=0} = \left.\frac{dy/dt}{dx/dt}\right|_{t=0} = \left.\frac{\frac{44}{10} - \frac{2}{10}t}{\frac{117}{22}}\right|_{t=0} = \frac{\frac{44}{10}}{\frac{117}{22}}$

$= \frac{44\cdot 22}{10\cdot 117} = \frac{968}{1170} < 1.$

$\boxed{\text{less than } 45^\circ}$

$22 \times 44 = 88 + 88 = 968$

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3. For this problem, use the differential equation for Newton's Law of Cooling:

$$\frac{dT}{dt} = k(T - T_s)\,,$$

where T is the temperature, t is time, T_s is the ambient temperature, and k is a constant.

Question: ① Water at temperature 50°F is placed in a freezer which is at 20°F. ② After 1 hour, the water is at temperature 40°F. ③ How long does it take the water to freeze, that is, to reach 32°F? (Partial credit for writing the general solution without solving for it.)

$$\frac{dT}{T-T_s} = k\,dt$$

$$\int \frac{dT}{T-T_s} = \int k\,dt$$

$$\ln|T-T_s| = kt + C$$

$$|T-T_s| = e^{kt+C}$$

$$T - T_s = Ae^{kt}$$

$$\boxed{T = T_s + Ae^{kt}}$$ gen'l soln

① info: $T_s = 20$, $T(0) = 50$

$$50 = 20 + Ae^{k\cdot 0}$$

$$30 = A$$

$$\boxed{T = 20 + 30e^{kt}}$$

② info: $T(1) = 40$

$$40 = 20 + 30e^{k\cdot 1}$$

$$20 = 30e^{k\cdot 1}$$

$$\tfrac{2}{3} = e^{k\cdot 1}$$

$$\ln\tfrac{2}{3} = k$$

$$\boxed{T = 20 + 30e^{(\ln\frac{2}{3})t}}$$ particular soln.

③ solve t with $T(t) = 32$

$$32 = 20 + 30e^{(\ln\frac{2}{3})t}$$

$$12 = 30e^{(\ln\frac{2}{3})t}$$

$$\tfrac{2}{5} = e^{(\ln\frac{2}{3})t}$$

$$\ln\tfrac{2}{5} = (\ln\tfrac{2}{3})t$$

$$\boxed{\frac{\ln\frac{2}{5}}{\ln\frac{2}{3}} = t}$$

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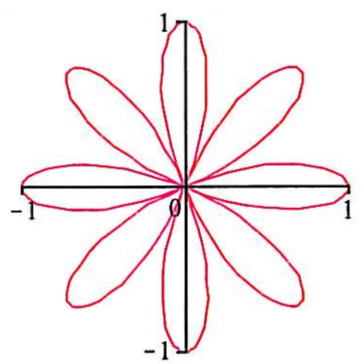
4. Find the area enclosed by one loop of the polar curve $r(\theta) = \cos(4\theta)$.

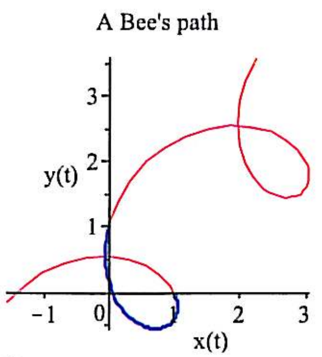


See other key

5. A bee buzzes along a path with parametric description

$$x(t) = \frac{t}{2} + \cos\left(\frac{\pi t}{2}\right),$$
$$y(t) = \frac{t}{2} - \sin\left(\frac{\pi t}{2}\right).$$

Set up (**but do not integrate**) an integral representing the arc length of the portion of the path starting at $(x_1, y_1) = (1, 0)$ and ending at $(x_2, y_2) = (0, 1)$. The integrand should be a function of t which contains no derivatives.



$AL = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

$x'(t) = \frac{1}{2} + \left(-\frac{\pi}{2}\right)\sin\left(\frac{\pi t}{2}\right)$

$y'(t) = \frac{1}{2} - \frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)$

$(x_2, y_2) = (0, 1)$

$x(t) = 0 = \frac{t}{2} + \cos\left(\frac{\pi t}{2}\right)$

$$\boxed{AL = \int_0^2 \sqrt{\left(\frac{1}{2} - \frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right)\right)^2 + \left(\frac{1}{2} - \frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)\right)^2}\, dt}$$

$(x_1, y_1) = (1, 0)$

$y(t) = 0 = \frac{t}{2} - \sin\left(\frac{\pi t}{2}\right)$

$t = 0$: $0 - \sin 0 = 0 = y_1$ ✓

$0 + \cos 0 = 1 = x_1$

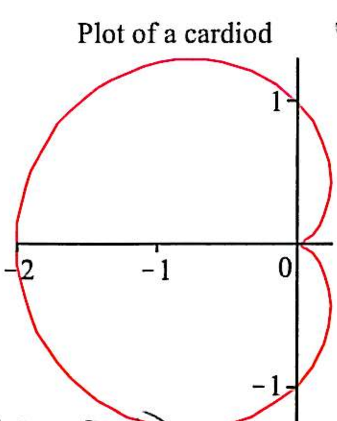
$t = 2$: $\frac{2}{2} + \cos\pi = 2 - 1 = 0 = x_2$

$\frac{2}{2} - \sin\pi = 1 = y_2$ ✓

6. A cardiod is given in polar form by the function $r(\theta) = 1 - \cos(\theta)$ for $0 \le \theta \le 2\pi$. Find the **horizontal** tangents to the plot of the cardiod, carefully describing the behavior at the cusp (at the pole). (Hint: It may be helpful to convert expressions to either all cosines or all sines.)

$x(\theta) = (1 - \cos\theta)\cos\theta = \cos\theta - \cos^2\theta$

$y(\theta) = (1 - \cos\theta)\sin\theta = \sin\theta - \sin\theta\cos\theta$



$\frac{dx}{d\theta} = -\sin\theta + 2\cos\theta\sin\theta = \sin\theta(2\cos\theta - 1)$

$\frac{dy}{d\theta} = \cos\theta - \cos\theta\cos\theta + \sin\theta\sin\theta = \cos\theta - \cos^2\theta + (1 - \cos^2\theta)$

$= \cos\theta - 2\cos^2\theta + 1 = -(2\cos\theta + 1)(\cos\theta - 1)$

$\frac{dy}{d\theta} = 0 = -(2\cos\theta + 1)(\cos\theta - 1)$

when $2\cos\theta + 1 = 0$ or $\cos\theta - 1 = 0$

$\cos\theta = -\frac{1}{2}$ $\cos\theta = 1$

$\theta = \boxed{\frac{2\pi}{3}, \frac{4\pi}{3}}$ $\boxed{\theta = 0}$

horiz tangent

Check L'Hospital

$\frac{dy}{dx} = \frac{0}{0}$ form

$\frac{dx}{d\theta} = 0 = \sin\theta(2\cos\theta - 1)$

when $\sin\theta = 0$ or $2\cos\theta - 1 = 0$

$\theta = 0, \pi$ $\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

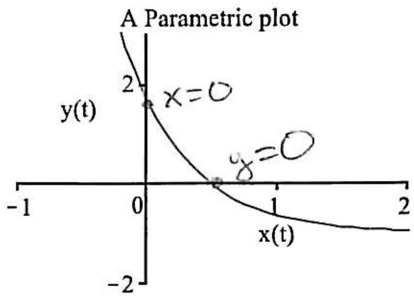
$$\lim_{\theta \to 0^+} \frac{dy}{dx} \overset{L'H}{\rightarrow} \lim_{\theta \to 0^+} \frac{-\sin\theta + 4\cos\theta\sin\theta}{-\cos\theta - 2\sin\theta\sin\theta - 2\cos\theta\cos\theta} = \frac{0}{-1-2}$$

horiz tangent approaching 0 from right.

$r(\theta) = r(-\theta)$, so ~~does~~ same approaching 0 from left.

7. Find the area in the first quadrant bounded between the origin and the parametric curve given by

$$
\begin{aligned}
x(t) &= t^2 - 1, \quad = f(t) \\
y(t) &= e^{-2t^2+3} - 1, \quad = g(t) \\
t &\geq 0.
\end{aligned}
\qquad (1)
$$



$$\text{Area} = \int_a^b g(t)\, f'(t)\, dt$$

$$f'(t) = 2t$$

$x(t) = 0 = t^2 - 1$

$1 = t^2$

$\pm 1 = t$

$\boxed{t = 1}$

$y(t) = 0 = \exp(-2t^2+3) - 1$

$1 = \exp(-2t^2+3)$

$0 = -2t^2 + 3$

$\frac{3}{2} = t^2$

$\pm\sqrt{\frac{3}{2}} = t$

$\boxed{t = \sqrt{\frac{3}{2}}}$

$$\text{Area} = \int_1^{\sqrt{\frac{3}{2}}} \left(e^{-2t^2+3} - 1\right)(2t)\, dt$$

$u = -2t^2 + 3$

$du = -4t\, dt$

$-\frac{du}{2} = 2t\, dt$

$$= \int_{t=1}^{t=\sqrt{\frac{3}{2}}} -\frac{(e^u - 1)}{2}\, du$$

$$= -\frac{e^u}{2} + \frac{u}{2} \Bigg|_{t=1}^{t=\sqrt{\frac{3}{2}}}$$

$$= -\frac{\exp(-2t^2+3)}{2} + \frac{(-2t^2+3)}{2} \Bigg|_1^{\sqrt{\frac{3}{2}}}$$

$$= \left(-\frac{\exp(0)}{2} + 0\right) - \left(-\frac{\exp(1)}{2} + \frac{1}{2}\right)$$

$$= \boxed{\frac{e}{2} - 1}$$