SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. Compute $\lim_{x \to \infty} (x \cdot e^x)^{1/x}$ IF ∞^0

$$= \lim_{x\to\infty} \exp\left(\ln\left[(xe^x)^{1/x}\right]\right)$$

$$= \exp\Big(\underbrace{\lim_{x\to\infty} \frac{1}{x} \ln[xe^x]}_{\text{IF } \frac{\infty}{\infty}}\Big)$$

$$\overset{L'H}{=} \exp\left(\lim_{x\to\infty} \frac{\frac{xe^x+e^x}{xe^x}}{1}\right)$$

$$= \exp\left(\lim_{x\to\infty} 1 + \frac{1}{x}\right)$$

$$= \exp(1) = \boxed{e}$$

limit exists, so we may write = here

2. (6pts) Evaluate $\int x^2 \cos(3x)\, dx$.

parts $u = x^2$ $dv = \cos 3x\, dx$

$du = 2x\,dx$ $v = \frac{1}{3}\sin 3x$

$$= \frac{x^2}{3}\sin 3x + \underbrace{\int -\frac{2x}{3}\sin 3x\, dx}_{\text{parts}}$$

$u = -\frac{2x}{3}$ $dv = \sin 3x\, dx$

$du = -\frac{2}{3}dx$ $v = -\frac{1}{3}\cos 3x$

$$= \frac{x^2}{3}\sin 3x + \frac{2x}{9}\cos 3x - \int \frac{2}{9}\cos 3x\, dx$$

$$= \boxed{\frac{x^2}{3}\sin 3x + \frac{2x}{9}\cos 3x - \frac{2}{27}\sin 3x + C}$$

3. (6pts) Evaluate $\int (\ln x)^2\, dx$. (Recall: $(\ln x)^2 \neq \ln(x^2)$).

parts $u = (\ln x)^2$ $dv = dx$

$du = 2(\ln x)\left(\frac{1}{x}\right)dx$ $v = x$

$$= x(\ln x)^2 + \underbrace{\int -2\ln x\, dx}_{\text{parts}}$$

$u = -2\ln x$ $dv = dx$

$du = -\frac{2}{x}dx$ $v = x$

$$= x(\ln x)^2 - 2x\ln x + \int 2\,dx = \boxed{x(\ln x)^2 - 2x\ln x + 2x + C}$$

4. Evaluate $\displaystyle\int_0^{\pi/2} \cos^3(7x)\,dx$.

$$= \int_0^{\frac{\pi}{2}} \cos 7x(\cos^2 7x)\,dx$$

$$= \int_0^{\pi/2} \cos 7x(1-\sin^2 7x)\,dx$$

u-sub: $u = \sin 7x$

$du = 7\cos 7x\,dx$

$\frac{du}{7} = \cos 7x\,dx$

$$= \frac{1}{7}\int_{x=0}^{x=\pi/2} (1-u^2)\,du$$

$$= \frac{1}{7}\left(u - \frac{u^3}{3}\right)\Big|_{x=0}^{x=\pi/2}$$

$$= \frac{1}{7}\left(\sin 7x - \frac{\sin^3 7x}{3}\right)\Big|_0^{\pi/2}$$

$$= \frac{1}{7}\left(-1 - \frac{(-1)^3}{3}\right) - \frac{1}{7}(0-0)$$

$$= -\frac{1}{7} + \frac{1}{21} = \boxed{\frac{-2}{21}}$$

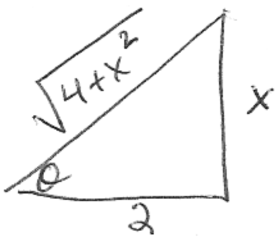
5. Use an appropriate trig substitution of the form $x = g(\theta)$ to reexpress $\displaystyle\int_0^{2\sqrt{3}} \frac{x^2}{\sqrt{4+x^2}}\,dx$ **completely** in terms of θ. **DO NOT** evaluate the resulting integral.

$x = 2\tan\theta, \ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 2\sec^2\theta\,d\theta$

$x^2 = 4\tan^2\theta$

$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta} = \sqrt{4\sec^2\theta} = 2\sec\theta$



$x = 0 = 2\tan\theta$

$0 = \tan\theta$

$\arctan 0 = \theta$

$\boxed{0 = \theta}$

$x = 2\sqrt{3} = 2\tan\theta$

$\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \tan\theta$

$\boxed{\theta = \frac{\pi}{3}}$

$$\int_{0}^{2\sqrt{3}} \frac{x^2}{\sqrt{4+x^2}}\,dx$$

$\frac{\pi}{3}$, 0, $4\tan^2\theta$, $2\sec^2\theta\,d\theta$, $2\sec\theta$

$\parallel$

$$\boxed{\begin{aligned}&\int_0^{\pi/3} \frac{4\tan^2\theta\sec^2\theta\,d\theta}{\sec\theta}\\ &= \int_0^{\pi/3} 4\tan^2\theta\sec\theta\,d\theta\end{aligned}}$$

6. Compute the partial fraction decomposition for $\dfrac{x^2+8x+9}{(x+1)(x+2)^2}$. (Do not integrate.)

$$\frac{x^2+8x+9}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} = \boxed{\frac{2}{x-1} + \frac{-1}{x+2} + \frac{3}{(x+2)^2}}$$

$$x^2+8x+9 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$$

$x=-1$: $1-8+9 = 2 = A\cdot 1 + B\cdot 0 + C\cdot 0$

$\boxed{A=2}$

$x=-2$: $4-16+9 = -3 = A\cdot 0 + B\cdot 0 + C(-1)$

$\boxed{C=3}$

$x^2+8x+9 = 2(x+2)^2 + B(x+1)(x+2) + 3(x+1)$ $\boxed{B=-1}$

x^2 term: $x^2 = 2x^2 + Bx^2 + C\cdot 0$

7. (16pts) **(a)** Find the general solution to the differential equation $\frac{1}{u}\frac{du}{dt} = k$, where k is a constant, and t is time. (Show steps.)

(b) From part **(a)**, find the particular solution satisfying $u(0) = 50$. Use this particular solution for parts **(c)** and **(d)**.

(c) Suppose k is positive. Find the time time T at which $u(T) = 100$.

(d) On the other hand, suppose $u(10) = 25$ and solve for k. Is k **positive or negative**?

(a) separate variables

$$\frac{du}{u} = k\,dt$$
$$\int \frac{du}{u} = \int k\,dt$$
$$\ln|u| = kt + C$$
$$|u| = e^{kt+C}$$
$$u = \pm e^{C} e^{kt}$$

$u=0$ also a solution

$$\boxed{u = Ae^{kt}}$$

(b)
$$u(0) = 50$$
$$A \cdot e^{k \cdot 0} = 50$$
$$A \cdot = 50$$
$$\boxed{u = 50e^{kt}}$$

(c)
$$u(T) = 100$$
$$50e^{kT} = 100$$
$$e^{kT} = 2$$
$$kT = \ln(2)$$
$$\boxed{T = \frac{\ln(2)}{k}}$$

(d)
$$u(10) = 25$$
$$50e^{k \cdot 10} = 25$$
$$e^{k \cdot 10} = \frac{1}{2}$$
$$k \cdot 10 = \ln\left(\tfrac{1}{2}\right)$$
$$\boxed{k = \frac{\ln(\frac{1}{2})}{10}}$$

$\ln\frac{1}{2} < 0 \Rightarrow$ $\boxed{k \text{ negative}}$

8. A population $P(t)$ at time t is modeled by the differential equation $\frac{dP}{dt} = 2.05P\left(1 - \frac{P}{4300}\right)$.

For this problem, ignore negative values of P ("population" is nonnegative).

(a) For what values of P is the population increasing?

(b) For what values of P is the population decreasing?

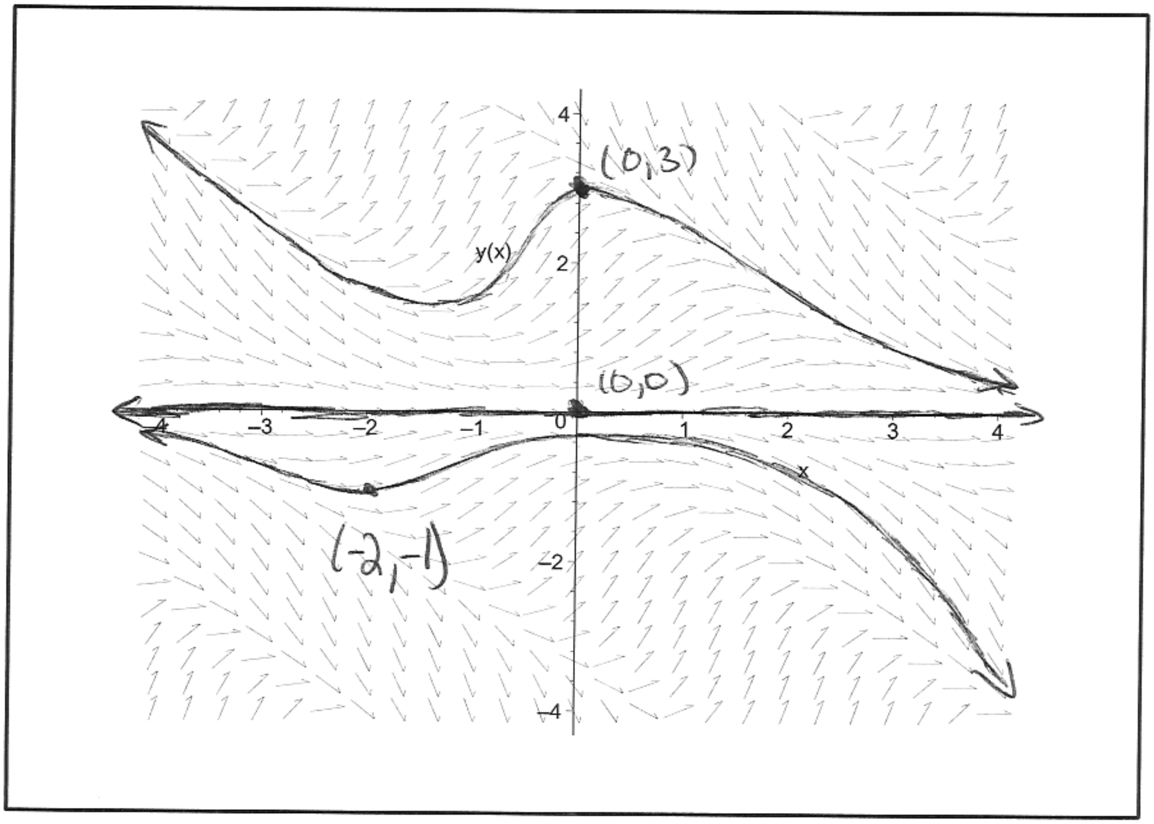
(c) What are the equilibrium solutions?

(a) P decr $\Leftrightarrow \frac{dP}{dt} < 0$ (b) P inc. $\Leftrightarrow \frac{dP}{dt} > 0$ (c) equilibrium $\Leftrightarrow \frac{dP}{dt} = 0$

	0		4300	
P	0	+	+	+
$\left(1 - \frac{P}{4300}\right)$	+	+	0	−
$2.05P\left(1 - \frac{P}{4300}\right)$	0	+	0	−

(a) $\boxed{\text{on } 0 < P < 4300}$ (b) on $P > 4300$ (c) $\boxed{P=0,\ P=4300}$

9. The direction field corresponding to a differential equation is given here. Carefully sketch 3 particular solutions, passing through the points $(0,0)$, $(0,3)$, and $(-2,-1)$, respectively.



10. (12pts) Determine whether $\displaystyle\int_1^\infty \frac{dx}{x^2+x+5}$ converges or diverges by comparing it to $\displaystyle\int_1^\infty \frac{dx}{x^p}$ for an appropriate value of p. (This requires computing the integral for your value of p).

set $p=2$.

$0 < x^2 < x^2+x+5$ on $[1,\infty)$

$\frac{1}{x^2} > \frac{1}{x^2+x+5} > 0$ on $[1,\infty)$

$\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}dx$

$= \lim_{t\to\infty} \left.\frac{x^{-1}}{-1}\right|_1^t$

$= \lim_{t\to\infty} \left(-\frac{1}{t}+1\right) = 1.$

Since $\int_1^\infty \frac{dx}{x^p}$ converges,

and $0 \le \frac{1}{x^2+x+5} \le \frac{1}{x^2}$

on $[1,\infty)$

then

$\boxed{\int_1^\infty \frac{dx}{x^2+x+5} \text{ converges by comparison.}}$