SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. Compute $\lim_{x\to 0^+} (\cos(2x))^{1/x}$ IF 1^∞

$= \lim_{x\to 0^+} \exp\left(\ln\left[(\cos 2x)^{1/x}\right]\right)$ cancellation exp, ln

$= \lim_{x\to 0^+} \exp\left(\frac{1}{x}\ln(\cos 2x)\right)$ rule 3 of logs

$= \exp\left(\lim_{x\to 0^+} \underbrace{\frac{\ln(\cos 2x)}{x}}\right)$ IF $\frac{0}{0}$ limit inside con't fxn

L'H $\overset{}{=} \exp\left(\lim_{x\to 0^+} \frac{\frac{-2\sin 2x}{\cos 2x}}{1}\right)$ apply L'Hospital

$= \exp\left(\lim_{x\to 0^+} -2\tan 2x\right)$ simplify

$= \exp(0)$ tanx is continuous at x = 0

$= \boxed{1}$

(limit exists, so now we may write ~~arrow~~ equals here)

2. (6pts) Evaluate $\int 3x^2 e^{-2x}\,dx$.

$3x^2$	e^{-2x}
$6x$	$-\frac{1}{2}e^{-2x}$
6 (-)	$\frac{1}{4}e^{-2x}$
	$-\frac{1}{8}e^{-2x}$

so $\int 3x^2 e^{-2x}\,dx$

$= \boxed{-\frac{3}{2}x^2 e^{-2x} - \frac{3x}{2}e^{-2x} - \frac{3}{4}e^{-2x} + C}$

-or-

$u = 3x^2 \quad dv = e^{-2x}dx$

$du = 6x\,dx \quad v = e^{-2x}/-2$

$= -\frac{3}{2}x^2 e^{-2x} + \underbrace{\int 3x e^{-2x}\,dx}_{\text{parts}}$

$u = 3x \quad dv = e^{-2x}dx$

$du = 3\,dx \quad v = e^{-2x}/-2$

$= -\frac{3}{2}x^2 e^{-2x} - \frac{3x}{2}e^{-2x} + \int \frac{3}{2} e^{-2x}\,dx$

$= \boxed{-\frac{3}{2}x^2 e^{-2x} - \frac{3}{2}x e^{-2x} - \frac{3}{4}e^{-2x} + C}$

3. (6pts) Evaluate $\int_0^{1/2} \arcsin x\,dx$.

$u = \arcsin x \quad dv = dx$

from cover $\left\{ du = \frac{1}{\sqrt{1-x^2}}dx \right. \quad v = x$

$= x\arcsin x\Big|_0^{\frac{1}{2}} - \underbrace{\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}dx}_{u\text{-sub}}$

$u = 1-x^2$

$du = -2x\,dx$

$\frac{du}{2} = -x\,dx$

$= x\arcsin x\Big|_0^{\frac{1}{2}} + \frac{1}{2}\int_{x=0}^{x=\frac{1}{2}} \frac{du}{\sqrt{u}}$

$= x\arcsin x + \frac{1}{2}\frac{u^{1/2}}{\frac{1}{2}}\Big|_{x=0}^{x=1/2} + \cancel{C}$

$= \boxed{x\arcsin x + \sqrt{1-x^2}}\Big|_0^{\frac{1}{2}} + \cancel{C}$

$= \left(\frac{1}{2}\cdot\frac{\pi}{6} + \sqrt{1-\frac{1}{4}}\right) - (0+1)$

$= \boxed{\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1}$

4. Evaluate $\int_0^{\pi/2} \sin^3(5x)\,dx$.

$$= \int_0^{\pi/2} \sin 5x \sin^2 5x \, dx$$

$$= \int_0^{\pi/2} \sin 5x \,(1-\cos^2 5x)\,dx$$

$u = \cos 5x$

$du = -5 \sin 5x \, dx$

$-\frac{du}{5} = \sin 5x \, dx$

$$= \int_{x=0}^{x=\pi/2} -\frac{(1-u^2)}{5}\,du$$

$$= -\frac{1}{5}\left(u - \frac{u^3}{3}\right)\Big|_{x=0}^{\pi/2}$$

$$= \frac{1}{5}\left(\frac{\cos^3 5x}{3} - \cos 5x\right)\Big|_0^{\pi/2}$$

$$= \frac{1}{5}\left(\frac{0^3}{3} - 0\right) - \frac{1}{5}\left(\frac{1^3}{3} - 1\right)$$

$$= \frac{1}{5} - \frac{1}{15} = \boxed{\frac{2}{15}}$$

5. Use an appropriate trig substitution of the form $x = g(\theta)$ to reexpress $\int_{3/2}^{3\sqrt{3}/2} \frac{\sqrt{9-x^2}}{x^2}\,dx$ **completely** in terms of θ. **DO NOT** evaluate the resulting integral.

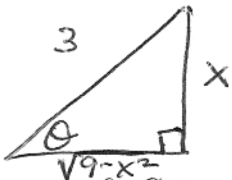
$x = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = 3\cos\theta \, d\theta$

$\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta}$

$= \sqrt{9\cos^2\theta}$

$= 3\cos\theta$



$x = \frac{3}{2} = 3\sin\theta$

$\frac{1}{2} = \sin\theta$

$\arcsin\frac{1}{2} = \theta$

$\frac{\pi}{6} = \theta$

$x = \frac{3\sqrt{3}}{2} = 3\sin\theta$

$\frac{\sqrt{3}}{2} = \sin\theta$

$\arcsin\frac{\sqrt{3}}{2} = \theta$

$\frac{\pi}{3} = \theta$

$$\int_{3/2}^{3\sqrt{3}/2} \frac{\sqrt{9-x^2}}{x^2}\,dx$$

$\frac{\pi}{3}$ $\quad$ $3\cos\theta$ $\quad$ $\pi/6$ $\quad$ $9\sin^2\theta$ $\quad$ $3\cos\theta\, d\theta$

$$\boxed{= \int_{\pi/6}^{\pi/3} \frac{9\cos^2\theta}{9\sin^2\theta}\,d\theta = \int_{\pi/6}^{\pi/3} \cot^2\theta \, d\theta}$$

6. Compute the partial fraction decomposition for $\frac{x^2+9x-16}{(x-4)(x-1)^2}$. (Do not integrate.)

$$\frac{x^2+9x-16}{(x-4)(x-1)^2} = \frac{A}{x-4} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \boxed{\frac{4}{x-4} + \frac{-3}{x-1} + \frac{2}{(x-1)^2}}$$

$x^2+9x-16 = A(x-1)^2 + B(x-4)(x-1) + C(x-4)$

$x=1$: $-6 = A\cdot 0 + B \cdot 0 + C(-3)$

$\boxed{C = 2}$

$x=4$: $16+36-16 = 36 = A\cdot 9 + B\cdot 0 + C\cdot 0$

$\boxed{A = 4}$

$x^2+9x-16 = 4(x-1)^2 + B(x-4)(x-1) + C(x-4)$

term: $x^2 = 4x^2 + Bx^2 + C\cdot 0$ $\quad$ $\boxed{B = -3}$

7. (16pts) **(a)** Find the general solution to the differential equation $\frac{1}{y}\frac{dy}{dt} = k$, where k is a constant and t is time. (Show steps.)

(b) From part **(a)**, find the particular solution satisfying $y(0) = 100$. Use this particular solution for parts **(c)** and **(d)**.

(c) Suppose k is negative. Find the time time T at which $y(T) = 50$.

(d) On the other hand, suppose $y(10) = 150$ and solve for k. Is k **positive or negative**?

(a) $\frac{dy}{dt} = yk$

separate variables:

$\frac{dy}{y} = k\,dt$

$\int \frac{dy}{y} = \int k\,dt$

$\ln|y| = kt + C$

$|y| = e^{kt+C}$

$|y| = e^{kt}e^{C}$

$y = \pm e^{C}e^{kt}$

$y=0$ also a solution, so

$\boxed{y = Ae^{kt}}$

(b) $y(0) = 100$

$Ae^{k\cdot 0} = 100$

$A \cdot 1 = 100$

$A = 100$

$\boxed{y = 100e^{kt}}$

(c) $y(T) = 50$

$100e^{kT} = 50$

$e^{kT} = \frac{1}{2}$

$kT = \ln\frac{1}{2}$

$\boxed{T = \frac{\ln\frac{1}{2}}{k}}$

(d) $y(10) = 150$

$100e^{k\cdot 10} = 150$

$e^{k\cdot 10} = \frac{3}{2}$

$k\cdot 10 = \ln\frac{3}{2}$

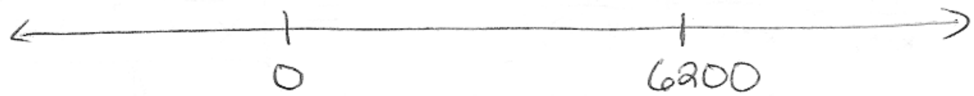
$\boxed{k = \frac{\ln\frac{3}{2}}{10}}$

$\ln\frac{3}{2} > 0$ so $\boxed{k \text{ is positive}}$

8. A population $P(t)$ at time t is modeled by the differential equation $\frac{dP}{dt} = 1.15P\left(1 - \frac{P}{6200}\right)$.
For this problem, ignore negative values of P ("population" is nonnegative).

(a) For what values of P is the population increasing?
(b) For what values of P is the population decreasing?
(c) What are the equilibrium solutions?

(a) corresponds to $\frac{dP}{dt} > 0$ (b) corresp. to $\frac{dP}{dt} < 0$ (c) corresp to $\frac{dP}{dt} = 0$



$P=0$ $P > 0$ $P > 0$

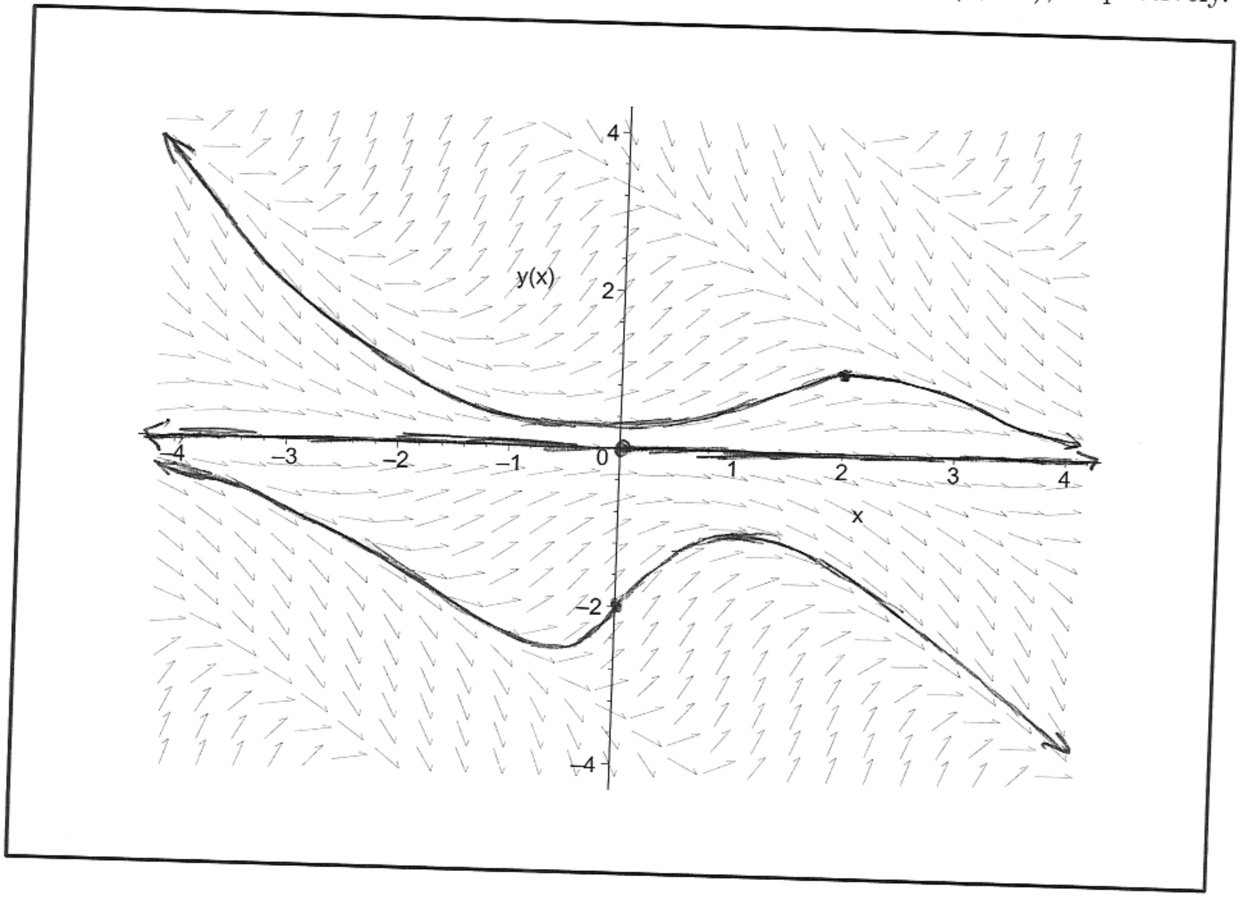
$1 - \frac{P}{6200} > 0$ $=0$ $1 - \frac{P}{6200} < 0$

(a) $\boxed{0 < P < 6200}$

(b) $\boxed{P > 6200}$

(c) $\boxed{P = 0,\ P = 6200}$

9. The direction field corresponding to a differential equation is given here. Carefully sketch 3 particular solutions, passing through the points $(0,0)$, $(2,1)$, and $(0,-2)$, respectively.



10. (12pts) Determine whether $\int_1^\infty \frac{dx}{x^{1/2} - \frac{1}{2}x^{1/4}}$ converges or diverges by comparing it to $\int_1^\infty \frac{dx}{x^p}$ for an appropriate value of p. (This requires computing the integral for your value of p).

set $p = \frac{1}{2}$.

$0 < +\frac{1}{2}x^{\frac{1}{4}} < x^{\frac{1}{2}}$ on $[1,\infty)$.

so $x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{4}} < x^{\frac{1}{2}}$ on $[1,\infty)$

and $\frac{1}{x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{4}}} > \frac{1}{x^{\frac{1}{2}}} \geq 0$ on $[1,\infty)$.

$\int_1^\infty \frac{1}{x^{\frac{1}{2}}} dx = \lim_{t\to\infty} \int_1^t x^{-\frac{1}{2}} dx$

$= \lim_{t\to\infty} 2x^{\frac{1}{2}}\Big|_1^t = \lim_{t\to\infty} (2\sqrt{t} - 2) = \infty$.

The smaller integral $\int_1^\infty \frac{dx}{x^{1/2}}$ diverges to ∞;

therefore the larger integral $\int_1^\infty \frac{dx}{x^{1/2} - \frac{1}{2}x^{1/4}}$ also diverges to ∞ by comparison