

SHOW WORK FOR FULL CREDIT

NO CALCULATORS

1. Find the inverse function, including domain and range, for the function  $f(x) = \sqrt{x-4}$  with domain  $4 \leq x \leq 13$ .

range of  $f$ :  $4 \leq x \leq 13$   
 $0 \leq x-4 \leq 9$   
 $0 \leq \sqrt{x-4} \leq 3$

range =  $[0, 3]$ .

so domain of  $f^{-1}$  is  $[0, 3]$   
 range of  $f^{-1}$  is  $[4, 13]$

find  $f^{-1}$ :  $y = \sqrt{x-4}$   
 $x = \sqrt{y-4}$   
 $x^2 = y-4$   
 $x^2 + 4 = y$

$f^{-1}(x) = x^2 + 4$

2. In order to stop the doomsday countdown of a computer on a certain island out of B. F. Skinner's nightmares, you must type in the derivative of a Lambert-type function  $g(x)$  at  $x = e$ . All you know is that  $g(x)$  is the inverse function of  $f(x) = x \cdot \ln x$ , where the domain of  $f$  is  $1 \leq x < \infty$ . Save the island by computing  $g'(e)$ .

cover formula:  $g'(e) = \frac{1}{f'(g(e))}$

$g(e) = x \Leftrightarrow f(x) = e$

solve  $x \cdot \ln x = e$   
 gives  $x = e$ .  
 thus  $g(e) = e$ .

$f'(x) = x(\frac{1}{x}) + 1 \cdot \ln x = 1 + \ln x$

so  $g'(e) = \frac{1}{f'(e)} = \frac{1}{1 + \ln e}$   
 $= \frac{1}{2}$

3. If  $f(x) = \frac{x-4}{x+4}$  and  $g(x)$  is the inverse of  $f(x)$ , find  $g(0)$ .

$g(0) = x \Leftrightarrow f(x) = 0$

solve  $\frac{x-4}{x+4} = 0$  for  $x$

$x-4 = 0$  ( $x+4$ )

$x = 4$

so  $g(0) = 4$

4. Let  $f(x) = \frac{1}{\sqrt{4-2x}}$ .

(a) What is the largest possible domain for  $f(x)$ ? (Show calculation.)

(b) Prove that  $f(x)$  is one-to-one on this domain by considering  $f'(x)$ . (Half-credit for a correct sketch plus the horizontal line test.)

(a) need  $4-2x > 0$  due to square root and reciprocal.

$-2x > -4$   
 $x < 2$

so domain is  $(-\infty, 2)$ .

when  $x < 2$ ,  $4-2x > 0$ ,  
 and  $(4-2x)^{-3/2} > 0$ .

so  $f'(x) > 0$  on domain,  
 $f$  is always increasing,  
 and thus is 1-1.

(b)  $f'(x) = -\frac{1}{2}(4-2x)^{-3/2}(-2)$   
 $= (4-2x)^{-3/2}$

5. Simplify the expressions (a)  $\cos\left(\arcsin\frac{1}{2}\right)$  and (b)  $\arccos\left(\cos\frac{8\pi}{6}\right)$ .

$$(a) \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\cos\left(\frac{\pi}{6}\right) = \boxed{\frac{\sqrt{3}}{2}}$$

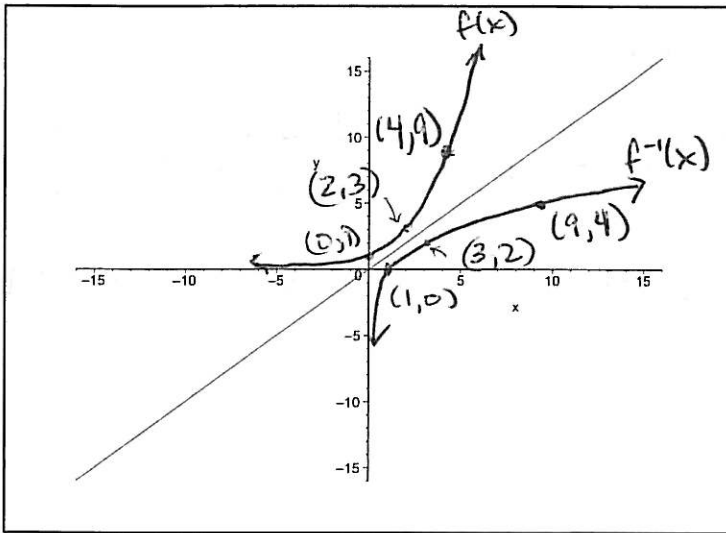
$$(b) \cos\left(\frac{8\pi}{6}\right) = -\frac{1}{2}$$

$$\cos^{-1}\left(-\frac{1}{2}\right) = \boxed{-\frac{\pi}{3}}$$

6. Let  $f(x) = (\sqrt{3})^x$ .

(a) What is the inverse of  $f(x)$ ?  $\text{E}$

(b) Plot and label  $f(x)$  and its inverse on the axes below, and label at least three pairs of points (6 total), where a pair of points has one point on the graph of  $y = f(x)$  and the corresponding point on the graph of the inverse.



$$(a) f^{-1}(x) = \log_{\sqrt{3}} x$$

$$(b) f(0) = 1$$

$$f(2) = 3$$

$$f(4) = 9$$

7. Evaluate the integral  $\int_0^{\pi/2} \sin x \cdot 3^{\cos x} dx$ .

$$u = \cos x$$

$$du = -\sin x dx$$

$$-du = \sin x dx$$

$$= \int_{x=0}^{x=\pi/2} -3^u du$$

$$= -\frac{3^u}{\ln 3} \Big|_{x=0}^{x=\pi/2}$$

$$= \frac{-3^{\cos x}}{\ln 3} \Big|_0^{\pi/2}$$

$$= \frac{-3^0}{\ln 3} - \left( \frac{-3^1}{\ln 3} \right)$$

$$= \boxed{\frac{2}{\ln 3}}$$

8. Differentiate the function  $f(x) = x \cdot e^{\sin(x)}$ .

$$f'(x) = x e^{\sin x} \cos x + 1 (e^{\sin x})$$

$$= e^{\sin x} (x \cos x + 1)$$

(product Rule)

9. Compute  $\lim_{x \rightarrow \infty} (x^2 + x)e^{-x}$ . form:  $\infty \cdot 0$

$$= \lim_{x \rightarrow \infty} \frac{x^2 + x}{e^x} \quad \text{form } \frac{\infty}{\infty} \text{ IF}$$

$$\xrightarrow{\text{L'H}} \lim_{x \rightarrow \infty} \frac{2x + 1}{e^x} \quad \text{form } \frac{\infty}{\infty} \text{ IF}$$

$$\xrightarrow{\text{L'H}} \lim_{x \rightarrow \infty} \frac{2}{e^x} = \lim_{x \rightarrow \infty} 2e^{-x} = 2 \lim_{x \rightarrow \infty} e^{-x} = 0.$$

therefore the original limit is  $\boxed{0}$

10. Compute  $\lim_{x \rightarrow 0^+} \frac{\arctan(x)}{x}$ . form:  $\frac{0}{0}$

$$\xrightarrow{\text{L'H}} \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x^2}}{1} = \lim_{x \rightarrow 0^+} \frac{1}{1+x^2} = \frac{1}{1 + \lim_{x \rightarrow 0^+} x^2} = \frac{1}{1}$$

therefore the original limit is  $\boxed{1}$

11. Compute  $\lim_{x \rightarrow \frac{\pi}{2}^-} (\tan x)^{\left(\frac{2x}{\pi}\right)}$ . form:  $\infty^1$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \exp(\ln(\tan x)^{\frac{2x}{\pi}}) = \lim_{x \rightarrow \frac{\pi}{2}^-} \exp\left(\frac{2x}{\pi} \ln \tan x\right)$$

$$= \exp\left(\lim_{x \rightarrow \frac{\pi}{2}^-} \underbrace{\frac{2x}{\pi} \ln \tan x}_{\text{Form } 1 \cdot \infty}\right) = \exp\left(\lim_{x \rightarrow \infty} x\right) = \lim_{x \rightarrow \infty} \exp(x) = \boxed{\infty}$$

12. Compute the derivative of the function  $f(x) = (\arcsin x)^{2x}$ .

$$y = (\sin^{-1} x)^{2x}$$

$$\ln y = 2x \ln \sin^{-1} x$$

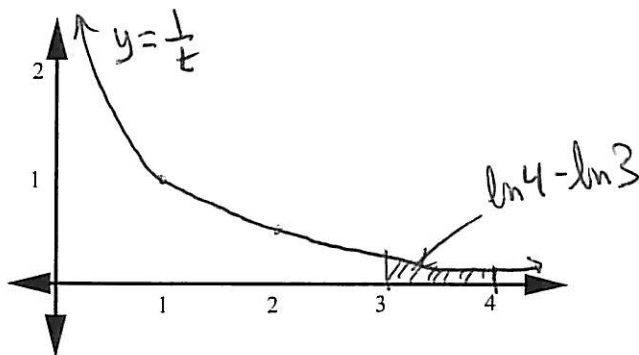
$$\frac{y'}{y} = 2x \frac{\frac{1}{\sqrt{1-x^2}}}{\sin^{-1} x} + 2 \ln \sin^{-1} x = \frac{2x}{\sqrt{1-x^2} \sin^{-1} x} + 2 \ln \sin^{-1} x$$

$$y' = (\arcsin x)^{2x} \left( \frac{2x}{\sqrt{1-x^2} \sin^{-1} x} + 2 \ln \sin^{-1} x \right)$$

13. Evaluate the integral  $\int \frac{\sec^2 x}{\tan x} dx$ .

$$\begin{aligned}
 & u = \tan x \\
 & du = \sec^2 x dx \\
 & = \int \frac{du}{u} = \ln|u| + C \\
 & = \boxed{\ln|\tan x| + C}
 \end{aligned}$$

14. Recall that by definition,  $\ln x := \int_1^x \frac{1}{t} dt$ . On the axes below, illustrate  $\ln 4 - \ln 3$  as an area, and label all relevant quantities.



15. Using the Laws of Logarithms write the following quantity in the form  $\ln(\cdot)$  (natural log of a single argument):  $2 \ln \cos(x) + 5 \ln(x-4) - 10 \ln(z+w)$

$$\begin{aligned}
 & = \ln(\cos x)^2 + \ln(x-4)^5 - \ln(z+w)^{10} \\
 & = \boxed{\ln \left[ \frac{\cos^2 x (x-4)^5}{(z+w)^{10}} \right]}
 \end{aligned}$$

16. Differentiate the function  $f(x) = \frac{4}{3} (\ln x)^3$ .

$$\begin{aligned}
 f'(x) & = \frac{4}{3} \frac{d}{dx} (\ln x)^3 \\
 & = \frac{4}{3} \cdot 3 (\ln x)^2 \left( \frac{d}{dx} \ln x \right) \\
 & = \frac{4}{3} \cdot 3 (\ln x)^2 \cdot \frac{1}{x} \\
 & = \boxed{\frac{4 (\ln x)^2}{x}}
 \end{aligned}$$