A large class of functions can be approximated by a so-called *finite Fourier sum* of the form

$$a + \sum_{n=1}^{N} b_n \sin nx + \sum_{n=1}^{N} c_n \cos nx \;=\; a + b_1 \sin x + b_2 \sin 2x + \cdots + b_N \sin Nx + c_1 \cos x + c_2 \cos 2x + \cdots + c_N \cos Nx,$$

where N is a fixed nonnegative integer ($N = 0$ means there are no sin or cos terms). N is the *degree* of the finite Fourier sum approximation. Finite Fourier sums can be used effectively to approximate any continuous function with period 2π, or any of a larger class of functions with certain technical restrictions. Of course, the period of a function can be scaled by replacing $x \to \theta x$, and we might not be interested in the approximation outside of $[-\pi, \pi]$. However, in this worksheet we will consider using finite Fourier sums to approximate a function $f(x)$ satisfying:
(i) $f : [-\pi, \pi] \to \mathbb{R}$, and
(ii) f is continuous except for at most finitely many discontinuities.

Meaning of *Approximate*. In order to approximate one object with a second object, we need a notion of approximation *error*. For our purposes, we define the error of approximating the function $f(x)$ by the function $g(x)$ to be

$$\text{Error} \;=\; \int_{-\pi}^{\pi} \left(f(x) - g(x)\right)^2 \, dx. \tag{1}$$

(The quantity most used is the square root of this expression, but minimizing the square root is the same as minimizing the original, so we'll let it slide for now.) We employ this definition in the following steps:
(1) Choose a function $f(x) : [-\pi, \pi] \to \mathbb{R}$ to approximate and choose a fixed nonnegative integer N for the degree of the approximation.
(2) The finite Fourier sum of degree N of $f(x)$ is the function $g(x) = a + \sum_{n=1}^{N} b_n \sin nx + \sum_{n=1}^{N} c_n \cos nx$ which minimizes $\int_{-\pi}^{\pi} \left(f(x) - g(x)\right)^2 \, dx$.
(3) Choose $a, b_1, \ldots, b_N, c_1, \ldots, c_N$ to minimize the error.

Theorem 1 (Computing Fourier sum coefficients)**.** *Let $f(x)$ be a function to be approximated by the finite Fourier sum $g(x) = a + \sum_{n=1}^{N} b_n \sin nx + \sum_{n=1}^{N} c_n \cos nx$ as described above. Then the coefficients $a, b_1, \ldots, b_N, c_1, \ldots, c_N$ are given by*

$$\begin{aligned} a &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \qquad \text{for } 1 \le n \le N \\ c_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \qquad \text{for } 1 \le n \le N. \end{aligned}$$

In particular, the coefficient defined in this fashion **minimize** *the error in (1).*

Example 1 (The square wave)**.** We now consider a square wave function which takes on a value of -1 between $-\pi$ and 0, and 1 between 0 and π (this is a different domain than the square wave given in class). This function can be described in many ways, including the following:

$$\begin{aligned} f(x) &= (-1)^{|\lfloor x/\pi \rfloor|}, \qquad \text{or} \\ f(x) &= \begin{cases} -1 & \text{if } -\pi \le x < 0, \\ 1 & \text{if } 0 \le x \le \pi. \end{cases} \end{aligned} \tag{2}$$

Now suppose we want a degree 1 finite Fourier sum

$$g(x) \;=\; a + b \sin x + c \cos x$$

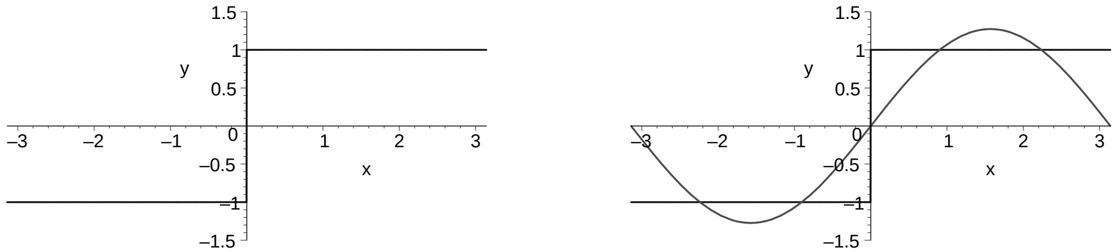

Figure 1: The square wave $f(x) = (-1)^{|\lfloor x\pi \rfloor|}$ on the left, along with its degree 1 finite Fourier sum approximation $g(x) = \frac{4}{\pi}\sin x$ on the right.

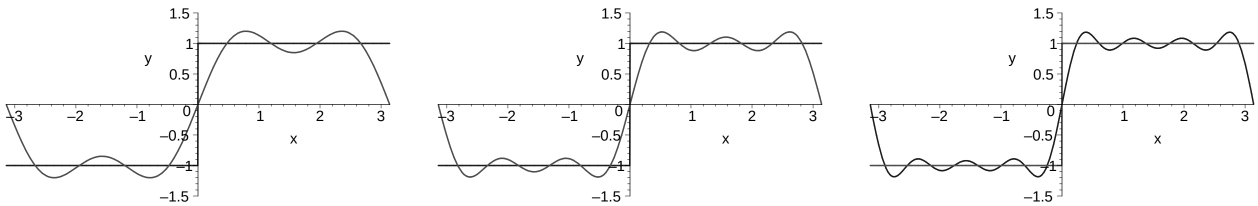

Figure 2: Finite Fourier sum approximations of the square wave of degrees 3, 5, and 7, respectively.

to approximate $f(x)$. We need to compute a, b, and c by using Theorem 1. We have

$$\begin{aligned}
a &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} -1\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 1\,dx = 0. \\
b &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\sin x\,dx = \frac{1}{\pi}\left(\cos x\Big|_{-\pi}^{0} - \cos x\Big|_{0}^{\pi}\right) = \frac{4}{\pi}. \\
c &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\cos x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cos x\,dx = \frac{1}{\pi}\left(-\sin x\Big|_{-\pi}^{0} \sin x\Big|_{0}^{\pi}\right) = 0,
\end{aligned}$$

Giving the finite Fourier sum approximation of degree 1 $g(x) = \frac{4}{\pi}\sin x$ of the square wave $f(x)$ (see Figure 1. We interpret the coefficients a, b, c as follows. Using $a \neq 0$ would not help the approximation because the square wave is symmetric about the x-axis, and is equally occuring at 1 as at -1. It makes sense that $b \neq 0$ because $f(x)$ is an odd function, as is $\sin x$, and both are either positive or negative simultaneously. Finally, $c = 0$ because $\cos x$ is an even function which doesn't reflect the behavior of the square wave at all. For instance, on the interval $[0, \pi]$ the square wave is always 1, but $\cos x$ is equally distributed to positive and negative values on that interval (however well $\cos x$ might approximate $f(x)$ on $[0, \pi/2]$, it would do equally poorly on $[\pi/2, \pi]$).

Maple Lab 2 Exercises, due 10/10/06, 10am

1. Use Maple to trace through and reproduce Example 1, where $f(x)$ is the square wave of period 2π.
   (a) Use Theorem 1 to show that the degree 3 finite Fourier sum approximation of the square wave is $g(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x$.
   (b) Find the error in using the degree 3 approximation.
   (c) Plot the square wave versus its degree 3 approximation.
   (d-f) Repeat (a-c) with the degree 5 finite Fourier sum approximation.

2. This problem also uses the square wave $f(x)$ of period 2π defined above.
(a) Explain why all of the cosine terms in a finite Fourier sum approximation of the square wave will have coefficient *zero*. Now for the rest of this problem, ignore the cosine terms in approximating $f(x)$.
(b) Find the finite Fourier sum $g(x)$ of smallest possible degree which approximates the square wave to within an error of 3/10. (*Hint:* use Maple and `evalf`.)

3. Verify directly that the choice of b which minimizes the error in approximating the square wave by $g(x) = b \sin x$ is $b = \frac{4}{\pi}$. (*Hint:* use Maple to exactly compute the error in (1) in terms of b.)

4. Repeat question 1 using a (nontrivial) waveform of your choice. Examples are available on the internet (e.g., http://en.wikipedia.org/wiki/Waveform,
http://www.answers.com/topic/waveform,
http://whatis.techtarget.com/definition/0,,sid9_gci213731,00.html).

Extra credit lab, due 10/10/06, 10am

Note: This lab is worth the same as all other Maple labs, but will count as extra credit (added to lab score rather than averaged in). No late submissions will be accepted. Partial credit will be awarded sparingly for these questions. Full credit equates to around 1.5 points on final grade.

1. For this problem, let $f(x) = e^x$.
(a) Let $g(x) = A + B \sin x + C \cos x$ and find the values of A, B and C which minimize equation (1) directly, and give the value of this minimum error. Describe why you can do this minimization even though three variables are involved.
(b) Compute A, B and C directly using Theorem 1 of the notes. Plot $f(x)$ and $g(x)$ on $-2\pi \le x \le 2\pi$ with a suitable range to view the behavior of f and g on this interval.
(c) Compute the degree 5 finite Fourier sum approximation $g(x)$ of $f(x)$, and plot them together on $-2\pi \le x \le 2\pi$ with a suitable range. Compute the error in the degree 5 approximation.
(d) Explain why any finite Fourier sum approximation of $f(x)$ will be bad outside of the interval $[-\pi, \pi]$.

Notes. You may find the maple command $\mathtt{sum}(h(i), i = 1..N)$ helpful in defining $g(x)$. For any "explanation" you should insert regular text into the worksheet. It is helpful to **resize plots** so they don't take up as much room when printing.

2. For this problem let $f(x)$ be the sawtooth wave function

$$f(x) \;=\; -\frac{x - 2\pi \left\lfloor \frac{x+\pi}{2\pi} \right\rfloor}{\pi}.$$

This is the function with period 2π, $f(-\pi) = 1$, $f(\pi) = -1$; and on $[-\pi, \pi]$, $f(x)$ is a line with slope $-1/\pi$. Now, find the smallest degree finite Fourier sum approximation $g(x)$ of the sawtooth wave so that the error is at most 1/4. Be sure to show $g(x)$, the degree which first makes the error below 1/4, and a plot of $f(x)$ and $g(x)$ with suitable domain and range.

3. Find a function $f(x)$ so that its finite Fourier sum approximation $g(x)$ satisfies the following properties:
(i) $f(x)$ is not equal to $g(x)$, and
(ii) $g(x)$ has no $\sin nx$ terms, regardless of the degree of $g(x)$ (do a calculation to verify this).
Plot $f(x)$ versus its degree 5 finite Fourier sum approximation for an appropriate doman and range.

Notes. For (ii), you may find the command `assume(n::posint);` useful, which tells Maple to assume that n is a positive integer.

Appendix 1: Useful commands

Definite integral: `int(f(x),x=a..b);`
Conversion of an expression to a function: `f1:=x->unapply(expression,x);`
Differentiation of function: `diff(f(x),x);`
Solving an equation involving functions: `solve(f(x)=g(x),x);`
Solving an equation involving expressions: `solve(expr1=expr2,x);`
Getting a numerical approximation of an exact number: `evalf(number);`
Plotting a function: `plot(f,g,-2*Pi..2*Pi,y=-10..20);`
or `plot(f(x),g(x),x=-2*Pi..2*Pi,y=-10..20);`

Defining a summation: `sum(f(i),i=startvalue..endvalue);`
For example, we might want to define a sum of sine term approximations of $f(x)$ by using the commands
`FSine := n -> (1/Pi)*int(f(x)*sin(n*x),x=-Pi..Pi);` (these are the coefficients $b_1, b_2, ...$)
`SineApprox := sum(FSine(i)*sin(i*x),i=1..5);.`
`SineApprox` is now the degree 5 approximation of $f(x)$ using sine terms only.

Appendix 2: Trigonometric integrals related to finite Fourier sums

Suppose we wish to verify the statement that any function of the form $f(x) = d + \sum_{n=1}^{N} e_n \sin nx + \sum_{n=1}^{N} f_n \cos nx$ is its own best degree N finite Fourier sum approximation? Clearly $f(x)$ is a degree N finite Fourier sum, and the error $\int_{-\pi}^{\pi} (f(x) - f(x))^2 \, dx = 0$. But suppose we instead want to verify this fact by using the rules to derive the coefficients a, b_n, c_n in Theorem 1 by showing $a = d$, $b_n = e_n$, and $c_n = f_n$? This will require computing integrals such as

$$a = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad \text{and} \quad c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

Breaking up these integrals into integrals corresponding to the $2N+1$ terms of $f(x)$, we end up with integrands which up to a constant are such as

$$\sin nx, \quad \cos nx, \quad \sin nx \sin mx, \quad \sin nx \cos mx, \quad \text{and} \quad \cos nx \cos mx.$$

Most of these integrals on $[-\pi, \pi]$ will be 0. For example, in computing $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$, all but the integral $\frac{1}{\pi}\int_{-\pi}^{\pi} e_n \sin nx \sin nx\,dx$ vanish, and this integral is e_n.

Suggested trigonometric integral exercises. The exercises which show the other integrands described above vanish are #59-#61 on p. 471. Additionally, #62 is essentially the problem of showing that $f(x) = d + \sum_{n=1}^{N} e_n \sin nx + \sum_{n=1}^{N} f_n \cos nx$ is its own best degree N finite Fourier sum approximation.