SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. For a certain series $\sum_{n=1}^{\infty} a_n$, the sequence of partial sums is

$$\{s_n\}_{n=1}^{\infty} = \{1 - 3^{-n}\}_{n=1}^{\infty}.$$

(a) (4pts) Find a closed formula for a_n.
(b) (4pts) Determine whether or not the series converges, and if so, find the sum.

(a) For $n \geq 2$, $a_n = s_n - s_{n-1} = (1-3^{-n}) - (1-3^{-(n-1)}) = -\frac{1}{3^n} + \frac{1}{3^{n-1}} = \frac{-1+3}{3^n} = \frac{2}{3^n}$

For $n=1$, $a_1 = s_1 = 1-3^{-1} = \frac{2}{3} = \frac{2}{3^1}$

The forms match. For all n, $a_n = \frac{2}{3^n}$.

(b) $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} (1-3^{-n}) = \lim_{x\to\infty} (1-3^{-x}) = \boxed{1}$

2. **(a) (5pts)** For what values of q does the series $\sum_{n=1}^{\infty} n^q$ converge?

(b) (5pts) For what values of s does the series $\sum_{n=0}^{\infty} \frac{7 \cdot s^n}{3^n}$ converge?

(a) $\sum_{n=1}^{\infty} n^q = \sum_{n=1}^{\infty} \frac{1}{n^{-q}}$ converges when $-q > 1$

or $q < -1$.

(b) geometric series.

First term 7

ratio $\frac{s}{3}$

converges when $\left|\frac{s}{3}\right| < 1$

$-1 < \frac{s}{3} < 1$

$\boxed{-3 < s < 3}$

3. **(10pts)** Starting with power series representation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{when } -1 < x < 1,$$

find a power series representation for $\frac{1}{8-x^3}$ and determine for exactly which values of x the representation is valid.

$$\frac{1}{8-x^3} = \frac{1}{8}\,\frac{1}{1-\frac{x^3}{8}} = \frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{x^3}{8}\right)^n \quad \text{when } \left|\frac{x^3}{8}\right| < 1$$

$$= \frac{1}{8}\sum_{n=0}^{\infty}\frac{x^{3n}}{8^n} \quad \text{when } -1 < \frac{x^3}{8} < 1$$

$$= \sum_{n=0}^{\infty}\frac{x^{3n}}{8^{n+1}} \quad \text{when } -8 < x^3 < 8$$

$$\text{or } -\cancel{8} < x < \cancel{8}.$$

$$-2 < x < 2$$

4. **(10pts)** Determine for *exactly* which values of x the power series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n \cdot n}$ converges.

ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{3^{n+1}(n+1)}}{\frac{(x-2)^n}{3^n n}}\right| = \lim_{n\to\infty}\left|\frac{(x-2)\,n}{3(n+1)}\right|$$

$$= \left|\frac{x-2}{3}\right|$$

converges when $\left|\frac{x-2}{3}\right| < 1$

$$-1 < \frac{x-2}{3} < 1$$

$$-3 < x-2 < 3$$

$$-1 < x < 5.$$

endpoints

$x=-1$: $\sum_{n=1}^{\infty}\frac{(-3)^n}{3^n n}$

$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

converges by alt. ser. test

$x=5$: $\sum_{n=1}^{\infty}\frac{3^n}{3^n \cdot n}$

$= \sum_{n=1}^{\infty}\frac{1}{n}$

diverges, p-series, $p \le 1$.

therefore the power series converges when

$$\boxed{-1 \le x < 5}$$

5. **(32pts)** For each test and each series below, check exactly one box of the four possible choices. Recall that not all choices are valid for all tests. A series would fail the preconditions for the alternating series test, for example, if the the signs are not alternating. (No partial credit. You do not have to show work.)

	Divergence Test	**Integral Test**	**Geometric Series Test**	**Alternating Series Test**
$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$	☐ fails pre-conditions ☐ diverges ☒ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☐ fails pre-conditions ☐ diverges ☐ inconclusive ☒ converges
$\sum_{n=1}^{\infty} \frac{1}{n}$	☐ fails pre-conditions ☐ diverges ☒ inconclusive ☐ converges	☐ fails pre-conditions ☒ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges
$\sum_{n=1}^{\infty} \frac{(-4)^n}{3^n}$	☐ fails pre-conditions ☒ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☐ fails pre-conditions ☒ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges
$\sum_{n=1}^{\infty} \frac{5^{n+1}}{3 \cdot 7^n}$	☐ fails pre-conditions ☐ diverges ☒ inconclusive ☐ converges	☐ fails pre-conditions ☐ diverges ☐ inconclusive ☒ converges	☐ fails pre-conditions ☐ diverges ☐ inconclusive ☒ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges

6. **(10pts)** Given the function $f(x) = \dfrac{1}{(x-2)^2+1}$, we define the sequence $\{a_n\}_{n=k}^{\infty}$ by setting $a_n = f(n)$.

(a) Find k as small as possible so that $\{a_n\}_{n=k}^{\infty}$ is a *decreasing* sequence. (Hint:use $f'(x)$.)

(b) Is the sequence bounded? If so, give valid bounds. If not, compute $\lim_{n\to\infty} a_n$.

(a) $f'(x) = \frac{d}{dx}\left((x-2)^2+1\right)^{-1} = -1\left((x-2)^2+1\right)^{-2}(2(x-2)) = \dfrac{-2(x-2)}{((x-2)^2+1)^2}$

$f'(x) < 0$ when $-2(x-2) < 0$

$x - 2 > 0$

$x > 2$

$f'(x) = 0$ when $x = 2$,

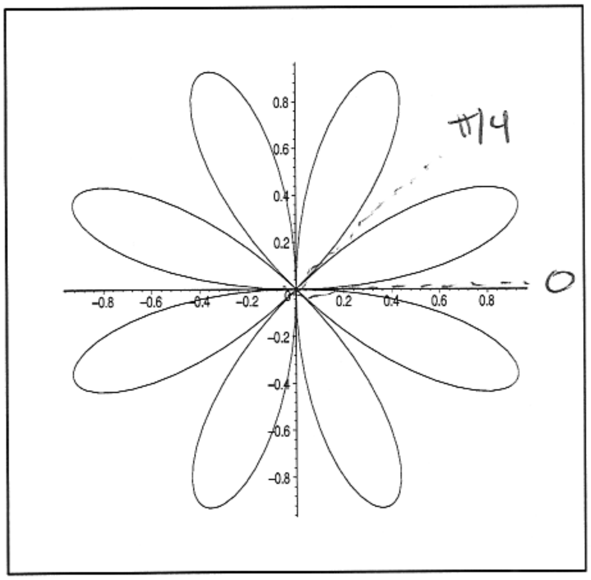
so $\boxed{k = 2}$

(b) $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{1}{(x-2)^2+1} = 0$

$a_2 = f(2) = 1$.

Sequence starts at 1 and decreases to 0, so it is bounded $\boxed{\text{above by } 1}$ and $\boxed{\text{below by } 0}$.

7. **(10pts)** Find the area enclosed by one loop of the polar curve $r(\theta) = \sin(4\theta)$.



$\sin 4\theta = 0$ when

$4\theta = 0, \pi, 2\pi, \ldots$

$\theta = 0, \frac{\pi}{4}, \ldots$

fmla: Area $= \int_\alpha^\beta \frac{1}{2}(f(\theta))^2 \, d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{1}{2} \sin^2(4\theta)\, d\theta$

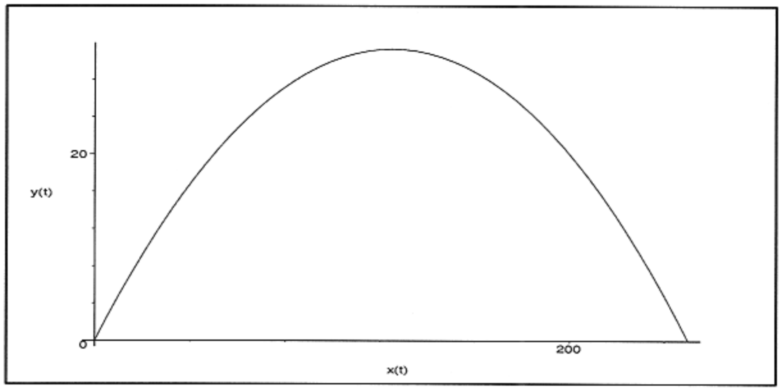
$= \int_0^{\frac{\pi}{4}} \frac{1}{2} \frac{1-\cos 8\theta}{2}\, d\theta$

$= \frac{\theta}{4} - \frac{\sin 8\theta}{32} \Big|_0^{\frac{\pi}{4}} = \left(\frac{\pi}{16} - 0\right) - (0 - 0) = \boxed{\frac{\pi}{16}}$

8. **(10pts)** A projectile is fired so that its horizontal and vertical position at time t in seconds is given by $x(t)$ and $y(t)$, respectively, where

$$\begin{aligned} x(t) &= 50t \\ y(t) &= 25t - 5t^2 \\ t &\geq 0. \end{aligned}$$

(a) Find A and B so that the projectile has height at least 20 exactly when $A \leq t \leq B$.
(b) Set up but **do not integrate** an integral representing the arc length traced out by the projectile when its height is at least 20. Write "A" and "B" in place of the quantities found in part (a).



(a) solve for $25t-5t^2 \geq 20$

$$5t^2-25t+20 \leq 0$$
$$t^2-5t+4 \leq 0$$
$$(t-1)(t-4) \leq 0$$

LHS is concave up (coeff of t^2 is >0) and has roots $t=1,4$.

Therefore $\boxed{A=1, \quad B=4}$

(b) $AL = \int_A^B \sqrt{(x'(t))^2+(y'(t))^2}\,dt$

$x'(t) = 50$
$(x'(t))^2 = 2500$
$y'(t) = 25-10t$
$(y'(t))^2 = (25-10t)^2$

$$\boxed{AL = \int_A^B \sqrt{2500+(25-10t)^2}\,dt}$$