SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. For a certain series $\sum_{n=1}^{\infty} a_n$, the sequence of partial sums is

$$\{s_n\}_{n=1}^{\infty} = \{1 - 4^{-n}\}_{n=1}^{\infty}.$$

(a) (4pts) Find a closed formula for a_n.
(b) (4pts) Determine whether or not the series converges, and if so, find the sum.

(a) for $n \geq 2$

$a_n = S_n - S_{n-1} = (1-4^{-n}) - (1-4^{-(n-1)}) = -\frac{1}{4^n} + \frac{1}{4^{n-1}} = \frac{-1+4}{4^n} = \frac{3}{4^n}$

for $n=1$: $a_1 = s_1 = 1 - 4^{-1} = \frac{3}{4} = \frac{3}{4^1}$

so $a_n = \frac{3}{4^n}$ for all $n \geq 1$.

(b) $\lim_{n\to\infty} S_n = \lim_{x\to\infty} (1-4^{-x}) = 1$, so $\sum_{n=1}^{\infty} a_n = 1$.

2. **(a) (5pts)** For what values of q does the series $\sum_{n=1}^{\infty} n^q$ converge?

(b) (5pts) For what values of s does the series $\sum_{n=0}^{\infty} \frac{5 \cdot 2^n}{s^n}$ converge?

(a) $\sum_{n=1}^{\infty} n^q = \sum_{n=1}^{\infty} \frac{1}{n^{-q}}$ Converges when $-q > 1$

or $q < -1$.

(b) Series is geometric.

First term 5

ratio $\frac{2}{s}$

converges for $\left|\frac{2}{s}\right| < 1$

$-1 < \frac{2}{s} < 1$

$-1 > \frac{s}{2} > 1$

$\boxed{-2 > s > 2}$

$s < -2$ and $s > 2$

3. **(10pts)** Starting with power series representation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{when } -1 < x < 1,$$

find a power series representation for $\dfrac{x^4}{1+x^3}$ and determine for exactly which values of x the representation is valid.

$$\frac{x^4}{1+x^3} = x^4 \cdot \frac{1}{1-(-x^3)} = x^4 \sum_{n=0}^{\infty} (-x^3)^n \quad \text{when } -1 < -x^3 < 1$$

$$= x^4 \sum_{n=0}^{\infty} (-1)^n x^{3n} \quad \text{when } 1 > x^3 > -1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{3n+4} \quad \text{when } +1 > x > -1$$

4. **(10pts)** Determine for *exactly* which values of x the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n}$ converges.

ratio test: $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty} \left| \frac{\frac{2^{n+1} x^{n+1}}{n+1}}{\frac{2^n x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{2xn}{n+1} \right|$$

$= |2x|$

Converges when $|2x| < 1$

$-1 < 2x < 1$

$-\frac{1}{2} < x < \frac{1}{2}$

endpoints

$x = -\frac{1}{2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (-\frac{1}{2})^n}{n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges by alt. series test

$x = \frac{1}{2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (\frac{1}{2})^n}{n}$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges by p-series, $p \le 1$

So convergence on $-\frac{1}{2} \le x < \frac{1}{2}$.

5. **(10pts)** For each test and each series below, check exactly one box of the four possible choices. Recall that not all choices are valid for all tests. A series would fail the preconditions for the alternating series test, for example, if the the signs are not alternating. (No partial credit. You do not have to show work.)

	Divergence Test	**Integral Test**	**Geometric Series Test**	**Alternating Series Test**
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$	☐ fails pre-conditions ☐ diverges ☒ inconclusive ☐ converges	☐ fails pre-conditions ☒ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges
$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$	☐ fails pre-conditions ☐ diverges ☒ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☐ fails pre-conditions ☐ diverges ☐ inconclusive ☒ converges
$\displaystyle\sum_{n=1}^{\infty} \frac{5 \cdot 4^{n-1}}{3^n}$	☐ fails pre-conditions ☒ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☐ fails pre-conditions ☒ diverges ☐ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges
$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{7^n}$	☐ fails pre-conditions ☐ diverges ☒ inconclusive ☐ converges	☒ fails pre-conditions ☐ diverges ☐ inconclusive ☐ converges	☐ fails pre-conditions ☐ diverges ☐ inconclusive ☒ converges	☐ fails pre-conditions ☐ diverges ☐ inconclusive ☒ converges

6. **(10pts)** Given the function $f(x) = \frac{x}{9} + \frac{1}{x}$, we define the sequence $\{a_n\}_{n=k}^{\infty}$ by setting $a_n = f(n)$.

(a) Find k as small as possible so that $\{a_n\}_{n=k}^{\infty}$ is an *increasing* sequence. (Hint:use $f'(x)$.)

(b) Is the sequence bounded? If so, give valid bounds. If not, compute $\lim_{n\to\infty} a_n$.

(a) $f'(x) = \frac{1}{9} - \frac{1}{x^2}$

$f'(x) > 0$

when $\frac{1}{9} - \frac{1}{x^2} > 0$

$\frac{1}{9} > \frac{1}{x^2}$

$9 < x^2$

$3 < x$

(ignore $x < -3$)

$f'(x) = 0$ at $x = 3$, so

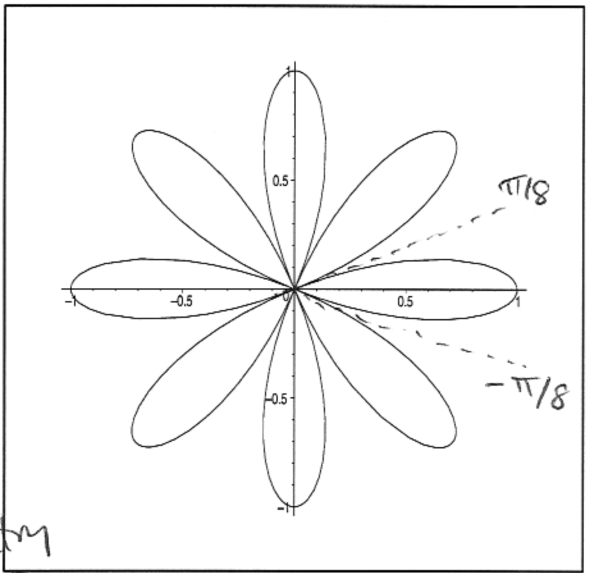
choose $\boxed{k = 3}$

(b) $\lim_{n\to\infty} a_n = \lim_{x\to\infty} \left(\frac{x}{9} + \frac{1}{x}\right)$

$= \infty$.

$\therefore$ not bounded.

7. **(10pts)** Find the area enclosed by one loop of the polar curve $r(\theta) = \cos(4\theta)$.



$\cos 4\theta = 0$

when $4\theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

or $\theta = \frac{-\pi}{8}, \frac{\pi}{8}, \ldots$

$\text{Area} = \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 \, d\theta$

$= \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{1}{2} \cos^2(4\theta) \, d\theta$

by symmetry about $\theta = 0$

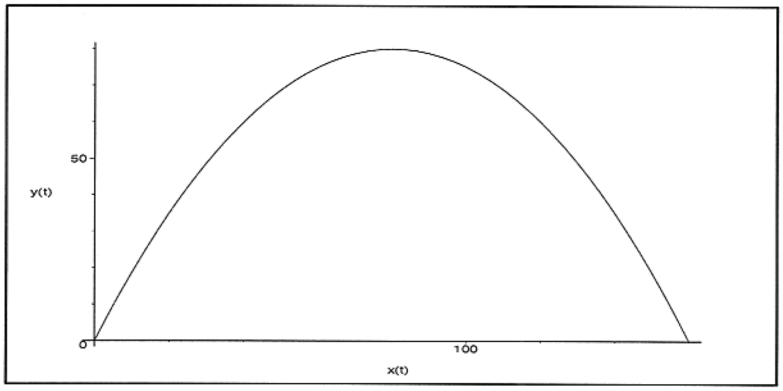
$= \int_0^{\frac{\pi}{8}} \cos^2(4\theta) d\theta = \int_0^{\frac{\pi}{8}} \frac{1 + \cos(8\theta)}{2} d\theta = \frac{\theta}{2} + \frac{\sin 8\theta}{16} \Big|_0^{\frac{\pi}{8}}$

$= \left(\frac{\pi}{16} + 0\right) - (0 + 0) = \boxed{\frac{\pi}{16}}$

8. **(10pts)** A projectile is fired so that its horizontal and vertical position at time t in seconds is given by $x(t)$ and $y(t)$, respectively, where

$$\begin{aligned} x(t) &= 20t \\ y(t) &= 40t - 5t^2 \\ t &\geq 0. \end{aligned}$$

(a) Find A and B so that the projectile has height at least 60 exactly when $A \leq t \leq B$.
(b) Set up but **do not integrate** an integral representing the arc length traced out by the projectile when its height is at least 60. Write "A" and "B" in place of the quantities found in part (a).



(a) solve $y(t) \geq 60$

$40t - 5t^2 \geq 60$

$5t^2 - 40t + 60 \leq 0$

$t^2 - 8t + 12 \leq 0$

$(t-6)(t-2) \leq 0$

coefficient of t^2 is >0, so function $(t-6)(t-2)$ is concave up with roots at $t=2$, $t=6$.

There fore $\boxed{A=2, \quad B=6.}$

(b) $AL = \int_A^B \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

$x'(t) = 20$

$(x'(t))^2 = 400$

$(y'(t)) = 40 - 10t$

$(y'(t))^2 = (40-10t)^2$

$AL = \int_A^B \sqrt{400 + (40-10t)^2}\, dt$