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B/

1. Find the general solution $y(x)$ of the differential equation

$$\frac{dy}{dx} = x\sqrt{1+x^2}(y+10).$$

Separable:

$$\frac{dy}{y+10} = x\sqrt{1+x^2}\,dx$$

$$\int \frac{dy}{y+10} = \int x\sqrt{1+x^2}\,dx$$

$u = 1+x^2$
$du = 2x\,dx$
$\int \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}u^{3/2}\frac{2}{3}+C = \frac{1}{3}(1+x^2)^{3/2}+C$

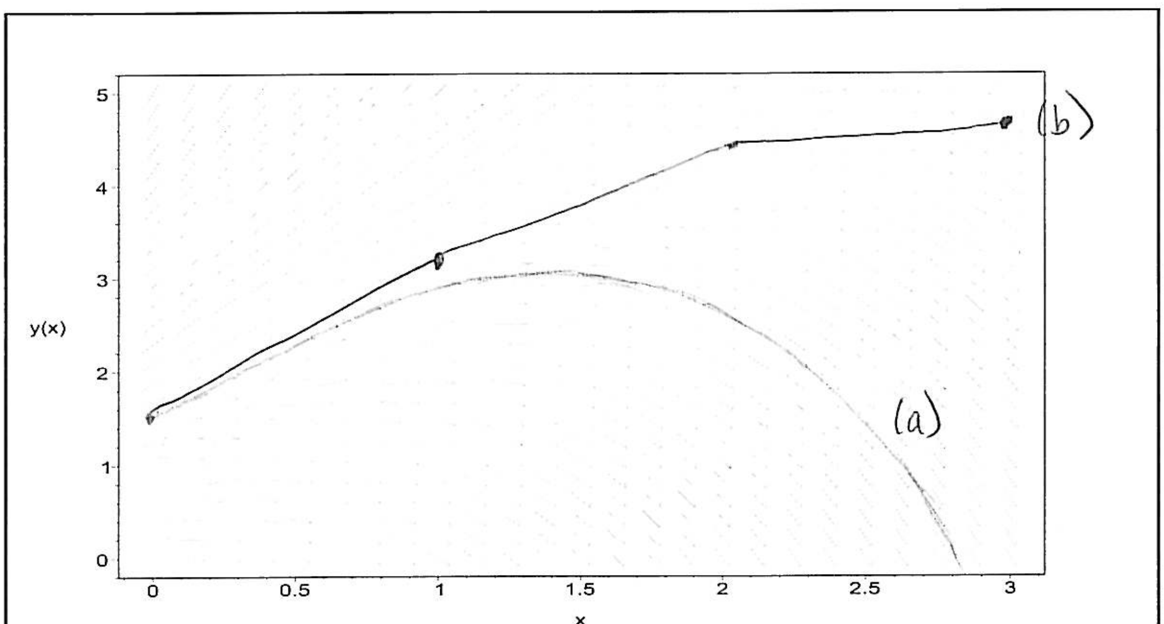
$$\ln|y+10| = \frac{1}{3}(1+x^2)^{3/2}+C$$

$$|y+10| = \exp\left(\frac{1}{3}(1+x^2)^{3/2}+C\right)$$

$$y+10 = A\exp\left(\frac{1}{3}(1+x^2)^{3/2}\right)$$

$$\boxed{y = A\cdot\exp\left(\frac{1}{3}(1+x^2)^{3/2}\right)-10}$$

2. The slope field of a differential equation is represented below.
(a) Trace the particular solution for the initial condition $y(0) = 1.5$.
(b) Use Euler's method with stepsize $\Delta x = 1$ to trace an approximation to the particular solution for the same initial condition.
(c) Make a general statement about how Euler's method will differ given the concavity of the particular solution.

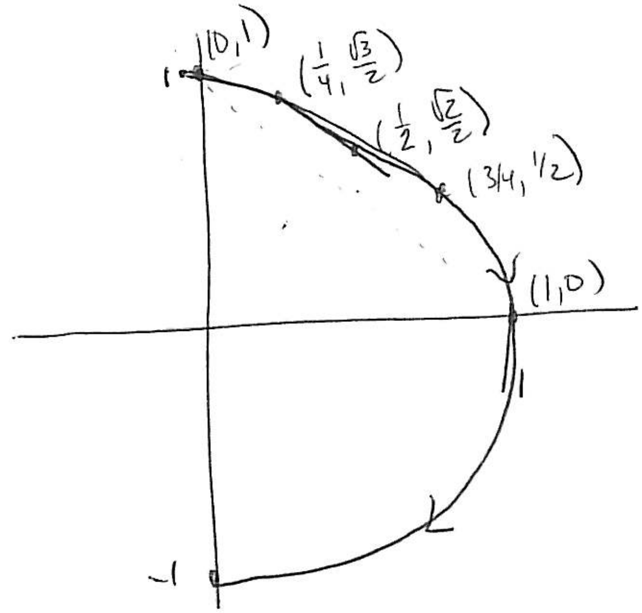


(c) since the particular solution is concave down, Eulers method will typically lie above the particular solution (not always, but true in this case)

3. Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

$$\begin{aligned} x(t) &= \cos^2 t \\ y(t) &= \sin t \\ \frac{\pi}{2} \leq\ & t \leq \frac{3\pi}{2} \end{aligned}$$

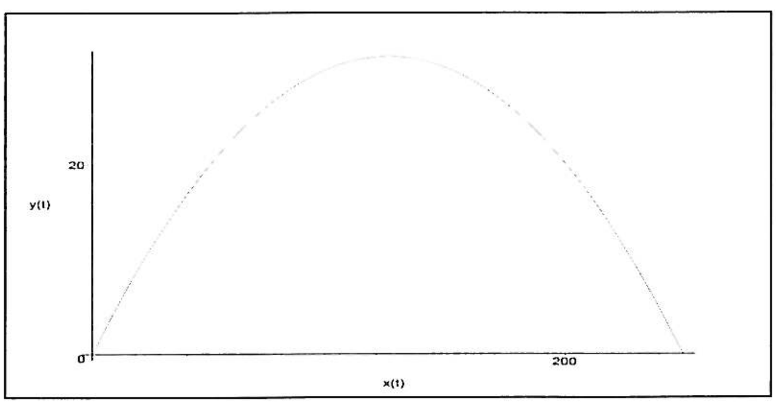
t	x	y
$\pi/2$	0	1
$\pi/3$	$1/4$	$\sqrt{3}/2$
$3\pi/4$	$1/2$	$\sqrt{2}/2$
$5\pi/6$	$3/4$	$1/2$
π	1	0
$\vdots$	these values in reverse	negative of these values in reverse



4. A projectile is fired so that its horizontal and vertical position at time t in seconds is given by $x(t)$ and $y(t)$, respectively, where

$$\begin{aligned} x(t) &= 50t \\ y(t) &= 25t - 5t^2 \\ t &\geq 0. \end{aligned}$$

(a) Find the maximum height (vertical position) of the projectile.
(b) Find the time at which this height is achieved.



find the horizontal tangent

$\frac{dy}{dt} = 0$

$25 - 10t = 0$

$10t = 25$

$t = \frac{5}{2}$

$y(\frac{5}{2}) = 25 \cdot \frac{5}{2} - 5 \cdot \left(\frac{5}{2}\right)^2 = \frac{125}{2} - \frac{125}{4} = \frac{125}{4}$

(a) $\frac{125}{4}$

(b) $t = \frac{5}{2}$ (seconds)

5. Find the value of D in the partial fraction decomposition

$$\frac{x^3 + x^2}{(x^2+3)^2} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{(x^2+3)^2}.$$

$A=1 \quad B=1$

$C=-3 \quad D=-3$

Clear denominator:

$x^3+x^2 = (Ax+B)(x^2+3) + Cx + D$

by inspection, $\cancel{x^3 = Ax \cdot x^2}$ $\quad x^2 = Bx^2$

$\cancel{1 = A}$ $\quad B = 1.$

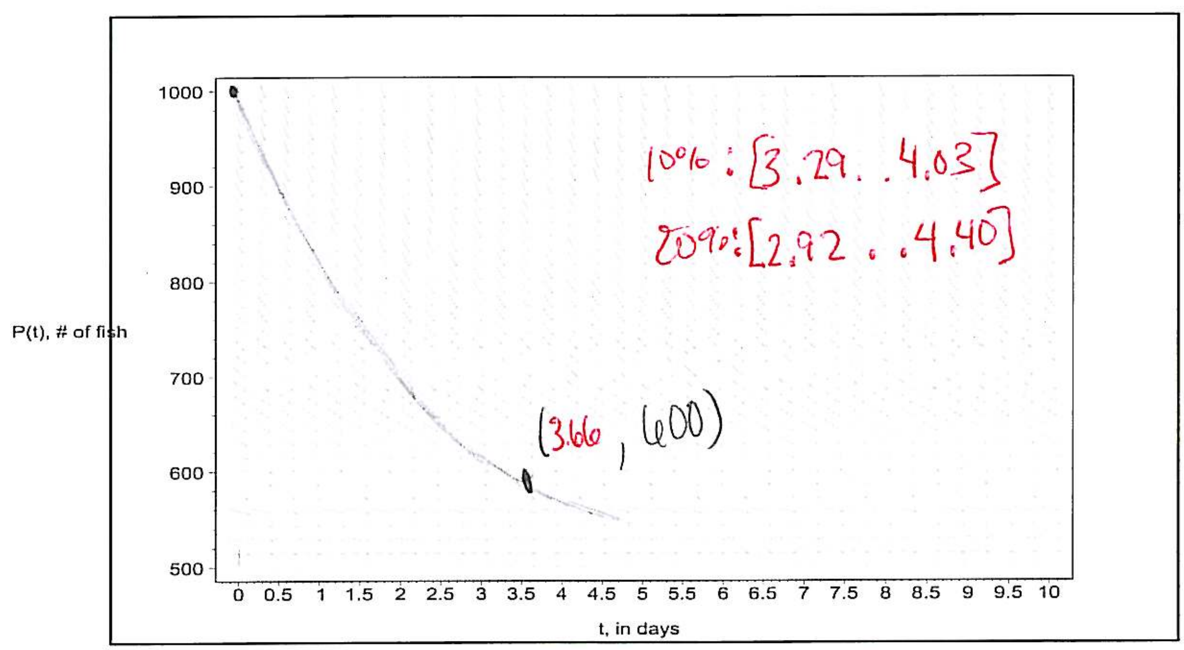
also, $\cancel{Ax \cdot 3 + Cx = 0 \cdot x}$ also, $0 \cdot x = B \cdot 3 + D$

$\cancel{3A + C = 0}$ $\quad -3B = D$

$\cancel{C = -3A}$ $\quad \boxed{-3 = D}$

$\cancel{\boxed{C = -3}}$

6. A fishery manager mistakenly puts 1000 fish in a pond which has carrying capacity only $K = 500$ fish. The following slope field is for the population $P(t)$ of fish at time t in days for the pond under the logistic growth model. Use the slope field to
(a) Estimate the time in days it takes for only 600 fish to remain (full credit for being correct within 10%); and
(b) Sketch the corresponding particular solution on the slope field.



7. The general solution to the ideal growth model for an investment with value $A(t)$ at time t in years can be written as

$$I(t) = P \cdot e^{k \cdot t}.$$

Suppose that after 1 year, the investment has increased in value by 10%.
(a) What is the doubling time of the investment, that is, the time it takes the investment to grow to double the initial value?
(b) What does P stand for?

$I(1) = 1.1\, I(0)$
$Pe^{k} = 1.1\, P$
$e^{k} = 1.1$
$k = \ln 1.1$

(b) P is the principal, or the initial investment at time $t=0$.

(a) solve $I(t) = 2\, I(0)$ for t
$Pe^{kt} = 2P$
$e^{\ln 1.1\, t} = 2$
$\ln 1.1\, t = \ln 2$
$\boxed{t = \dfrac{\ln 2}{\ln 1.1}}$

8. Given that

$$T(t) = 100 + A \cdot e^{-0.1 \cdot t}$$

is the general solution to a Newton's Law of Heating and Cooling differential equation:
(a) Find the particular solution given the initial condition $T(0) = 50$;
(b) Find the particular solution given the initial condition $T(0) = 150$; and
(c) Circle the particular solution which is **decreasing** when t is in the interval $[0, \infty)$.

(a) $T(0) = 50$
$100 + A = 50$
$A = -50$
$T(t) = 100 - 50e^{-.1t}$

(b) $T(0) = 150$
$100 + A = 150$
$A = 50$
$T(t) = 100 + 50e^{-.1t}$

(c)

decreasing on $[0,\infty)$.

9. Find and simplify the **integrating factor** for the following linear differential equation, which may or may not be in standard form (**do not** solve for the general solution):

$$y' \cdot \tan\theta + y \cdot \sec^2\theta = \theta^3, \qquad \text{where } 0 < \theta < \frac{\pi}{2}.$$

$$y' + y\frac{\sec^2\theta}{\tan\theta} = \frac{\theta^3}{\tan\theta}$$

$$I(\theta) = e^{\int \frac{\sec^2\theta}{\tan\theta}\,d\theta}$$

side: $\int \frac{\sec^2\theta}{\tan\theta}\,d\theta$

$u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$= \int \frac{du}{u} = \ln|u| + C$

$= \ln|\tan\theta| + C$

$= \ln(\tan\theta) + C$ since $\tan\theta > 0$ on $0 < \theta < \frac{\pi}{2}$

$$I(\theta) = e^{\ln\tan\theta}$$

$$\boxed{I(\theta) = \tan\theta}$$

10. Use the integrating factor $I(x) = \csc x$ to find the **general solution** of the differential equation

$$y' = x \cdot \sin^2 x + y \cdot \cot x.$$

$$y' - y\cot x = x\sin^2 x$$

$$y'\csc x - y\cot x\csc x = x\sin^2 x\csc x$$

$$(y\csc x)' = x\sin x$$

$$y\csc x = \int x\sin x\,dx$$

aside

x	$\sin x$
1	$-\cos x$
	$-\sin x$

$-x\cos x + \sin x + C$

$$y\csc x = -x\cos x + \sin x + C$$

$$\boxed{y = -x\cos x\sin x + \sin^2 x + C\sin x}$$