SHOW WORK FOR FULL CREDIT NO CALCULATORS A1

1. Find the value of C in the partial fraction decomposition

$$\frac{x^3+2x^2}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}.$$

clear denominator!

4 $x^3+2x^2 = (Ax+B)(x^2+2) + (Cx+D)$

by inspection, $x^3 = Ax \cdot x^2$

3 $1 = A$

also, $0 \cdot x = Ax \cdot 2 + Cx$

$0 = 2A + C$

3 $C = -2A$

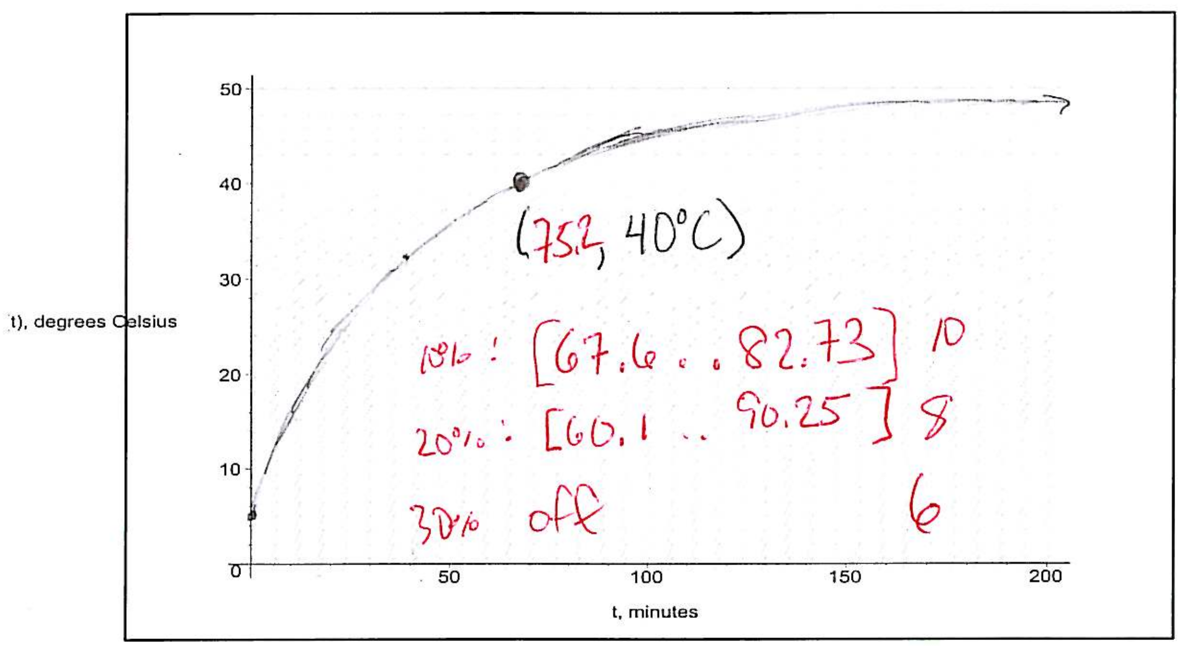
$\boxed{C = -2}$

$A = 1$ $B = 2$

$C = -2$ $D = -4$

careless error + no check: 8/10

2. The following slope field represents the Newton's Law of Heating and Cooling for an object with temperature $T(t)$ at time t for a certain environment with an ambient temperature of $T_S = 50°$C. Use the slope field to
(a) Estimate the time in minutes it takes for an object with temperature 5°C to increase to a temperature of 40°C when placed in this environment (full credit for being correct within 10%); and
(b) Sketch the corresponding particular solution on the slope field.



3. The general solution to the radioactive decay model for an amount $A(t)$ of a radioactive element at time t in years can be written as

$$A(t) = C \cdot e^{k \cdot t}.$$

Suppose that after 1 year, 20% of the original amount has decayed.
(a) What is the half-life of the element, that is, the time it takes for half of the original amount to decay?
(b) What does C stand for?

$A(1) = .8\,A(0)$

$Ce^{k} = .8\,C$

$e^{k} = .8$

$k = \ln .8$

(a) solve $A(t) = \frac{1}{2}A(0)$ for t

$Ce^{kt} = \frac{1}{2}C$

$e^{kt} = \frac{1}{2}$

$k = \ln .8$, so

$e^{\ln .8\, t} = \frac{1}{2}$

$\ln .8\; t = \ln \frac{1}{2}$

$\boxed{t = \dfrac{\ln \frac{1}{2}}{\ln .8}}$

(b) The initial amount of substance

4. Given that

$$P(t) = \frac{5000}{1 + A \cdot e^{-0.1 \cdot t}}$$

is the general solution to a logistic differential equation:
(a) Find the particular solution given the initial condition $P(0) = 2500$;
(b) Find the particular solution given the initial condition $P(0) = 7500$; and
(c) Circle the particular solution which is **increasing** when t is in the interval $[0, \infty)$.

(a) $P(0) = 2500$

$\frac{5000}{1+A} = 2500$

$2 = 1 + A$

$1 = A$

$P(t) = \frac{5000}{1 + e^{-.1t}}$

(b) $P(0) = 7500$

$\frac{5000}{1+A} = 7500$

$\frac{2}{3} = 1 + A$

$-\frac{1}{3} = A$

$P(t) = \frac{5000}{1 - \frac{1}{3}e^{-.1t}}$

(c) increasing.

(population < carrying capacity $\Rightarrow P'(t) > 0$)

5. Find and simplify the **integrating factor** for the following linear differential equation, which may or may not be in standard form (**do not** solve for the general solution):

$$y' \cdot \sin\theta + y \cdot \cos\theta = \theta^2, \qquad \text{where } 0 < \theta < \pi.$$

$$y' + y \cdot \frac{\cos\theta}{\sin\theta} = \frac{\theta^2}{\sin\theta}$$

$$I(\theta) = e^{\int \frac{\cos\theta}{\sin\theta} d\theta}$$

$$= e^{\ln \sin\theta}$$

$$I(\theta) = \boxed{\sin\theta}$$

aside: $\int \frac{\cos\theta}{\sin\theta} d\theta$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$\int \frac{du}{u} = \ln|u| + C$

$= \ln|\sin\theta| + C$

$= \ln \sin\theta + C$ since $\sin\theta > 0$ on $0 < \theta < \pi$.

6. Use the integrating factor $I(x) = \sec x$ to find the **general solution** of the differential equation

$$y' = x \cdot \cos^2 x - y \cdot \tan x.$$

$$y' + y \tan x = x \cos^2 x$$

$$y' \sec x + y \tan x \sec x = x \cos^2 x \sec x$$

$$(y \sec x)' = x \cos x$$

$$y \sec x = \int x \cos x \, dx$$

x	$\cos x$
1	$\sin x$
	$-\cos x$

$x \sin x + \cos x + C$

$$y \sec x = x \sin x + \cos x + C$$

$$\boxed{y = x \sin x \cos x + \cos^2 x + C \cos x}$$

7. Find the general solution $y(x)$ of the differential equation

$$\frac{dy}{dx} = \frac{x}{y^2 \cdot (x^2+5)}.$$

Separable:

$$y^2\,dy = \frac{x\,dx}{x^2+5}$$

$$\frac{y^3}{3} = \int \frac{x\,dx}{x^2+5}$$

aside

$$\int \frac{x\,dx}{x^2+5}$$

$u = x^2+5$

$du = 2x\,dx$

$$\int \frac{1}{2}\frac{du}{u} = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+5| + C$$

$$= \frac{1}{2}\ln(x^2+5) + C$$

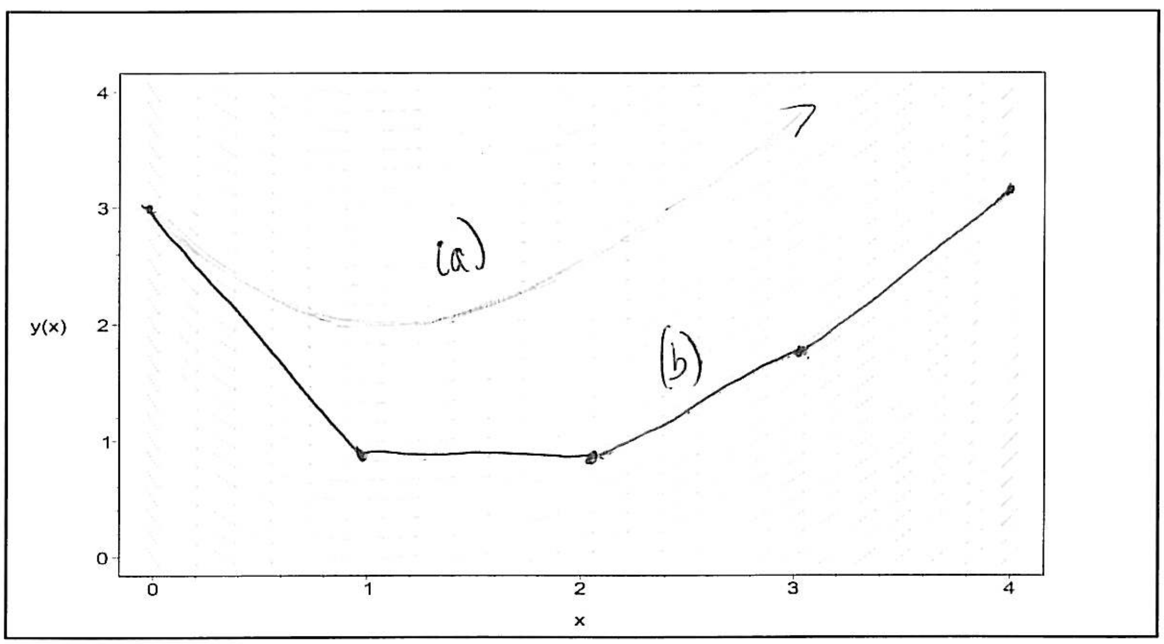
$$\frac{y^3}{3} = \frac{1}{2}\ln(x^2+5) + C$$

$$y^3 = \frac{3}{2}\ln(x^2+5) + C'$$

$$\boxed{y = \sqrt[3]{\tfrac{3}{2}\ln(x^2+5) + C'}}$$

(C' any constant)

8. The slope field of a differential equation is represented below.
(a) Trace the particular solution for the initial condition $y(0) = 3$.
(b) Use Euler's method with stepsize $\Delta x = 1$ to trace an approximation to the particular solution for the same initial condition.
(c) Make a general statement about how Euler's method will differ given the concavity of the particular solution.



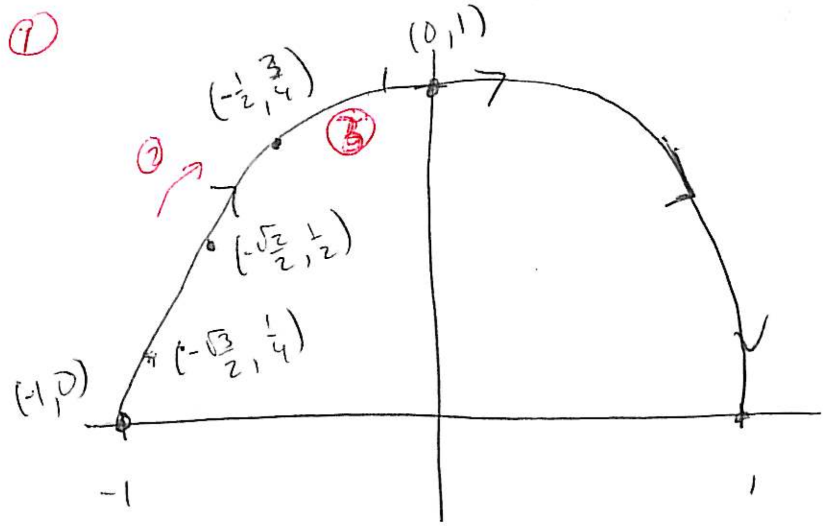
(c) since it is concave up, Euler's method will give an underestimate. This is typically true, although there are counterexamples with some slope fields.

9. Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

$$
\begin{aligned}
x(t) &= \cos t \\
y(t) &= \sin^2 t = 1 - x^2 \quad (1) \\
\pi \le\ & t \le 2\pi
\end{aligned}
$$

(4)

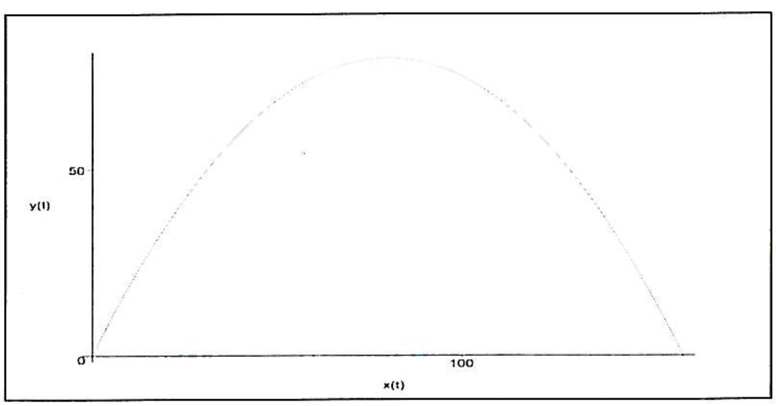
t	x	y
π	-1	0
$7\pi/6$	$-\sqrt{3}/2$	$1/4$
$5\pi/4$	$-\sqrt{2}/2$	$1/2$
$4\pi/3$	$-1/2$	$3/4$
$3\pi/2$	0	1
⋮	negative of these values, reversed	these values, reversed



10. A projectile is fired so that its horizontal and vertical position at time t in seconds is given by $x(t)$ and $y(t)$, respectively, where

$$
\begin{aligned}
x(t) &= 20t \\
y(t) &= 40t - 5t^2 \\
t &\ge 0.
\end{aligned}
$$

(a) Find the maximum height (vertical position) of the projectile.
(b) Find the time at which this height is achieved.



Find horizontal tangent.

$$\frac{dy}{dt} = 0$$

$$40 - 10t = 0$$

$$10t = 40$$

$$t = 4.$$

(a) $y(4) = 40 \cdot 4 - 5 \cdot 16 = 160 - 80 = \boxed{80}$

(b) $\boxed{t = 4}$ (seconds)