SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. Find A, B, and C in the partial fraction decomposition

$$\frac{2x^2-3x+7}{(x-1)(x^2+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+5}.$$

$2x^2-3x+7 = A(x^2+5)+(Bx+C)(x-1)$

$x=1$: $6 = A\cdot 6 + (Bx+C)\cdot 0$

$A = 1$

$2x^2-3x+7 = x^2+5+(Bx+C)(x-1)$

$x^2-3x+2 = (Bx+C)(x-1)$

by inspecting coefficient of x^2 and x^0,

$x^2 = Bx\cdot x$ $2 = C(-1)$

$1 = B$ $C = -2$

$$\boxed{\begin{array}{l} A=1 \\ B=1 \\ C=-2 \end{array}}$$

2. Evaluate the integral $\displaystyle\int \frac{2x-1}{(x+1)^2}\,dx$.

$\dfrac{2x-1}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2}$

$2x-1 = A(x+1)+B$

$x=-1$: $-3 = A\cdot 0 + B$

$-3 = B$

$2x-1 = A(x+1)-3$

$2x+2 = A(x+1)$

$2 = A$

$\displaystyle\int \frac{2x-1}{(x+1)^2}dx = \int\left(\frac{2}{x+1} + \frac{-3}{(x+1)^2}\right)dx$

$= \boxed{2\ln|x+1| + \dfrac{3}{x+1} + C}$

3. Find a function $f(x)$ such that $\displaystyle\int_1^\infty f(x)\sqrt{x}\,dx = \infty$, but $\displaystyle\int_1^\infty f(x)\,dx$ converges. Give the exact value of the second integral or otherwise prove it converges.

note that $\displaystyle\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{dx}{x} = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty}\ln t = \infty.$

set $\boxed{f(x) = \dfrac{1}{x\sqrt{x}} = \dfrac{1}{x^{3/2}}}$. So $\displaystyle\int_1^\infty f(x)\sqrt{x}\,dx = \int_1^\infty \frac{1}{x}dx = \infty$

but $\displaystyle\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^{3/2}}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^{3/2}}dx = \lim_{t\to\infty} \frac{x^{-1/2}}{-1/2}\Big|_1^t$

$\displaystyle = \lim_{t\to\infty}\left(\frac{t^{-1/2}}{-1/2} - \frac{1}{-1/2}\right) = \lim_{t\to\infty}\left(-\frac{2}{\sqrt{t}}+2\right) = \boxed{2}$ (converges to 2)

4. Use integration by parts to evaluate the integral $\int \sin^{-1} x \, dx$.

$u = \sin^{-1}x \qquad dv = dx$

$du = \dfrac{dx}{\sqrt{1-x^2}} \qquad v = x$

$\int \sin^{-1}x\,dx = x\sin^{-1}x - \underbrace{\int \frac{x}{\sqrt{1-x^2}}dx}$

$= \boxed{x \sin^{-1}x + \sqrt{1-x^2} + C}$

side problem

$\int \frac{x}{\sqrt{1-x^2}}dx \qquad u = 1-x^2, \quad du = -2x\,dx, \quad -\frac{du}{2} = x\,dx$

$= \int -\frac{1}{2}\frac{du}{\sqrt{u}}$

$= -\frac{1}{2}\frac{u^{1/2}}{1/2} + C$

$= -\sqrt{1-x^2} + C$

5. Recall that the degree 1 finite Fourier approximation of $f(x)$ is $g(x) = a + b\sin x + c\cos x$, where

$$a = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx\,, \quad b = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(x)\,dx\,, \text{ and } \quad c = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(x)\,dx\,.$$

When $f(x) = x^3$, determine whether a, b, and c are positive, negative, or 0, and **give a justification for each**. Circle one answer for each of a, b, and c.

a: $< 0 \quad = 0 \quad > 0$

way1: $f(x)$ is an odd function, so $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0$

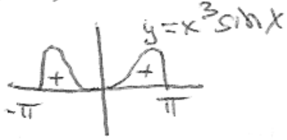
way2: $\frac{1}{2\pi}\int_{-\pi}^{\pi} x^3dx = \frac{1}{2\pi}\frac{x^4}{4}\Big|_{-\pi}^{\pi}$

$= \frac{1}{8\pi}\left(\pi^4 - (-\pi)^4\right)$

$= \frac{1}{8\pi}(0) = 0$

b: $< 0 \quad = 0 \quad > 0$

way1: $x^3\sin x > 0$ on $(0,\pi)$ and on $(-\pi, 0)$.



way2: $\frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin x\,dx$

$\frac{x^3}{\pi}$	$\sin x$
$3x^2/\pi$	$-\cos x$
$6x/\pi$	$-\sin x$
$6/\pi$	$\cos x$
	$\sin x$

$= -\frac{x^3}{\pi}\cos x + \frac{3x^2}{\pi}\sin x$

$+ \frac{6x}{\pi}\cos x - \frac{6}{\pi}\sin x\Big|_{-\pi}^{\pi}$

$= \left(-\pi^2(-1) + 0 - (-6 - 0)\right)$

$-(-\pi^2 + 0 + 6 - 0)$

$= 2\pi^2 - 12 > 2\cdot 9 - 12 = 6 > 0.$

c: $< 0 \quad = 0 \quad > 0$

way1: x^3 is an odd function, $\cos x$ is an even function so $x^3\cos x$ is an odd function and $\frac{1}{\pi}\int_{-\pi}^{\pi} x^3\cos x\,dx = 0$.

way2: $\frac{1}{\pi}\int_{-\pi}^{\pi} x^3\cos x\,dx$

x^3/π	$\cos x$
$3x^2/\pi$	$\sin x$
$6x/\pi$	$-\cos x$
$6/\pi$	$-\sin x$
	$\cos x$

$= \frac{x^3}{\pi}\sin x + \frac{3x^2}{\pi}\cos x - \frac{6x}{\pi}\sin x$

$-\frac{6}{\pi}\cos x\Big|_{-\pi}^{\pi}$

$= \left(0 - 3\pi - 0 + \frac{6}{\pi}\right)$

$-\left(0 - 3\pi - 0 + \frac{6}{\pi}\right) = 0.$

6. Evaluate the integral $\int \sec^4 x \tan^3 x \, dx$.

$\tan^2 x = \sec^2 x - 1$. reserve $\sec x \tan x \, dx$ for $u = \sec x$

$\int \sec^4 x \tan^3 x \, dx$

$= \int \sec^3 x (\sec^2 x - 1) \sec x \tan x \, dx$

$u = \sec x$

$du = \sec x \tan x \, dx$

$= \int u^3 (u^2 - 1) \, du$

$= \int (u^5 - u^3) \, du$

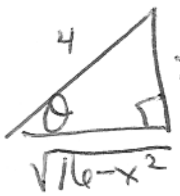
$= \frac{u^6}{6} - \frac{u^4}{4} + C$

$= \boxed{\frac{\sec^6 x}{6} - \frac{\sec^4 x}{4} + C}$

7. Evaluate the integral $\int_0^4 \sqrt{16 - x^2} \, dx$ in two ways: **(i)** by direct evaluation, and **(ii)** by interpreting the integral as a geometric area.

(i) $x = 4 \sin\theta$

$dx = 4 \cos\theta \, d\theta$



$\sin\theta = \frac{x}{4} = \frac{O}{\cancel{AH}}$

$\cos\theta = \frac{\sqrt{16-x^2}}{4}$

$4\cos\theta = \sqrt{16-x^2}$

$\theta = \sin^{-1}\left(\frac{x}{4}\right)$. $\sin 2\theta = \cancel{2\sin\theta}\cos$ $2\sin\theta\cos\theta$

$= 2 \cdot \frac{x}{4} \cdot \frac{\sqrt{16-x^2}}{4}$

$= \frac{x\sqrt{16-x^2}}{8}$

$\int_0^4 \sqrt{16-x^2} \, dx = \int_{x=0}^4 4\cos\theta \, 4\cos\theta \, d\theta$

$= 16 \int_{x=0}^4 \cos^2\theta \, d\theta$

$= 16 \int_{x=0}^4 \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$

$= 8\theta + 4\cancel{\sin}2\theta \Big|_{x=0}^4$

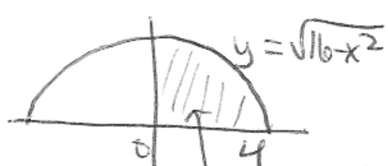
$8\theta + 4\sin 2\theta \Big|_{x=0}^4 =$

$8\sin^{-1}\left(\frac{x}{4}\right) + \frac{x\sqrt{16-x^2}}{2} \Big|_0^4$

$= (8\sin^{-1}(1) + 0) - (8\sin^{-1}(0) + 0)$

$= 8\frac{\pi}{2} - 0 = \boxed{4\pi}$

(ii)



area of $\frac{1}{4}$ circle of radius 4

$= \frac{1}{4} \cdot \pi (4)^2 = \boxed{4\pi}$

8. Find the exact value of the following:

(a) $\tan^{-1}(\tan(\pi))$

$\tan\pi = \frac{\sin\pi}{\cos\pi} = \frac{0}{-1} = 0.$

$f(x) = \tan^{-1}(x)$ has domain $(-\infty,\infty)$ and range $(-\frac{\pi}{2},\frac{\pi}{2})$.

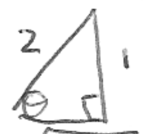
π is not in $(-\frac{\pi}{2},\frac{\pi}{2})$, but 0 is, and so

$\tan^{-1}(0) = \boxed{0}$

(b) $\cos(\sin^{-1}(1/2))$

let $\theta = \sin^{-1}(\frac{1}{2})$.

then $\sin\theta = \frac{1}{2}$ and $0 \le \theta \le \frac{\pi}{2}$



$\sqrt{4-1} = \sqrt{3}$

$\cos\theta = \boxed{\frac{\sqrt{3}}{2}}$

9. Evaluate the integral $\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$.

Let $u = \tan^{-1}x$

$du = \frac{1}{1+x^2}dx$

$\int \frac{\tan^{-1}(x)}{1+x^2}dx = \int u\,du = \frac{u^2}{2} + C$

$= \boxed{\frac{(\tan^{-1}(x))^2}{2} + C}$

10. Evaluate the integral $\int 3x^2 \sin(2x)\,dx$. $= \boxed{-\frac{3}{2}x^2\cos 2x + \frac{6x}{4}\sin 2x + \frac{6}{8}\cos 2x + C}$

$3x^2$	$\sin 2x$
$6x$	$-\frac{1}{2}\cos 2x$
6	$-\frac{1}{4}\sin 2x$
	$\frac{1}{8}\cos 2x$