SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. Find the exact value of the following:

(a) $\tan^{-1}(\tan(3\pi/4))$

$\tan\frac{3\pi}{4} = \frac{\sin 3\pi/4}{\cos 3\pi/4} = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = -1$

$\tan^{-1}(-1) = \boxed{\frac{-\pi}{4}}$

since $\frac{-\pi}{4}$ is in the range of $\tan^{-1}(x)$ but $\frac{3\pi}{4}$ is not.

(b) $\sin(\cos^{-1}(2/3))$

Let $\theta = \cos^{-1}(2/3)$.

Then $0 < \theta < \pi/2$


$\sqrt{9-4} = \sqrt{5}$

$\sin\theta = \frac{O}{H} = \boxed{\frac{\sqrt{5}}{3}}$

2. Evaluate the integral $\displaystyle\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$.

let $u = \sin^{-1}x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$

$= \int u\,du = \frac{u^2}{2} + C$

$= \boxed{\frac{(\sin^{-1}(x))^2}{2} + C}$

3. Evaluate the integral $\displaystyle\int 2x^2 e^{-3x}\,dx$. $= \boxed{-\frac{2}{3}x^2 e^{-3x} - \frac{4}{9}x e^{-3x} - \frac{4}{27}e^{-3x} + C}$

| | |
|---|---|
| $2x^2$ | $e^{-3x}$ |
| $4x$ | $-\frac{1}{3}e^{-3x}$ |
| $4$ | $\frac{1}{9}e^{-3x}$ |
| | $-\frac{1}{27}e^{-3x}$ |

4. Find $A$, $B$, and $C$ in the partial fraction decomposition

$$\frac{x^2+x+8}{(x-2)(x^2+3)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+3}.$$

$\boxed{\begin{array}{l} A=2 \\ B=-1 \\ C=-1 \end{array}}$

$x^2+x+8 = A(x^2+3) + (Bx+C)(x-2)$

$x=2$: $14 = A \cdot 7 + 0$

$\boxed{2=A}$

$x^2+x+8 = 2x^2+6 + (Bx+C)(x-2)$

$-x^2+x+2 = (Bx+C)(x-2)$

by inspection of coefficients of $x^2$, $x^0$ terms,

$-x^2 = Bx \cdot x$ $\quad$ $2 = C(-2)$

$\boxed{-1=B}$ $\quad$ $\boxed{-1=C}$

5. Evaluate the integral $\displaystyle\int \frac{x-1}{(x-2)^2}\,dx$.

$\displaystyle\int \frac{x-1}{(x-2)^2}dx = \int \frac{1}{x-2}dx + \int \frac{1}{(x-2)^2}dx$

$= \boxed{\ln|x-2| - \frac{1}{x-2} + C}$

$\displaystyle\frac{x-1}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$

$x-1 = A(x-2) + B$

$x-1 = Ax - 2A + B$

by inspection of $x^1$ term,

$\boxed{A=1}$.

$x-1 = x - 2 + B$

$\boxed{1=B}$

6. Find a function $f(x)$ such that $\displaystyle\int_1^\infty f(x)\,dx = \infty$, but $\displaystyle\int_1^\infty \frac{f(x)}{\sqrt{x}}\,dx$ converges. Give the exact value of the second integral or otherwise prove it converges.

~~Note that~~ $\displaystyle\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{dx}{x} = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln t = \infty.$

Let $\boxed{f(x)=\frac{1}{x}}$

so $\displaystyle\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}dx = \infty.$

but $\displaystyle\int_1^\infty \frac{f(x)}{\sqrt{x}}dx = \int_1^\infty \frac{1}{x^{3/2}}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^{3/2}}dx = \lim_{t\to\infty} \frac{x^{-1/2}}{-1/2}\Big|_1^t$

$\displaystyle= \lim_{t\to\infty}\left(\frac{t^{-1/2}}{-1/2} - \frac{1}{-1/2}\right) = 0+2 = \boxed{2}$

7. Use integration by parts to evaluate the integral $\displaystyle\int \tan^{-1} x\,dx$.

$u = \tan^{-1}x \qquad dv = dx$

$du = \frac{dx}{1+x^2} \qquad v = x$

$\displaystyle\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx$

$= \boxed{x\tan^{-1}x - \tfrac{1}{2}\ln(1+x^2) + C}$

side problem

$\displaystyle\int \frac{x}{1+x^2}\,dx \qquad u = 1+x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$= \frac{1}{2}\displaystyle\int \frac{du}{u}$

$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|1+x^2| + C$

$= \frac{1}{2}\ln(1+x^2) + C$

8. Recall that the degree 1 finite Fourier approximation of $f(x)$ is $g(x) = a + b\sin x + c\cos x$, where

$$a = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx\,, \quad b = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(x)\,dx\,, \text{ and } \quad c = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(x)\,dx\,.$$

When $f(x) = x^3$, determine whether $a$, $b$, and $c$ are positive, negative, or 0, and **give a justification for each**. Circle one answer for each of $a$, $b$, and $c$.

**a:** $< 0 \quad = 0 \quad > 0$ **b:** $< 0 \quad = 0 \quad > 0$ **c:** $< 0 \quad = 0 \quad > 0$

See Solution on Form B

9. Evaluate the integral $\int \sec^4 x \tan^2 x \, dx$.

reserve $\sec^2 x \, dx$ for $u = \tan x$, $du = \sec^2 x \, dx$

$\sec^2 x = 1 + \tan^2 x$

$$\int \sec^4 x \tan^2 x \, dx = \int (1+\tan^2 x)\tan^2 x \underbrace{\sec^2 x \, dx}_{du}$$

$$= \int (1+u^2) u^2 \, du = \int (u^2 + u^4) \, du$$

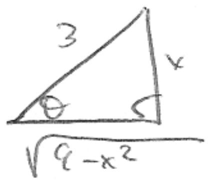
$$= \frac{u^3}{3} + \frac{u^5}{5} + C = \boxed{\frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C}$$

10. Evaluate the integral $\int_0^3 \sqrt{9 - x^2}\, dx$ in two ways: **(i)** by direct evaluation, and **(ii)** by interpreting the integral as a geometric area.

(i) $\int_0^3 \sqrt{9-x^2}\, dx$

$x = 3\sin\theta$

$dx = 3\cos\theta \, d\theta$




$\sin\theta = \frac{O}{H} = \frac{x}{3}$

$\cos\theta = \frac{A}{H} = \frac{\sqrt{9-x^2}}{3}$

$3\cos\theta = \sqrt{9-x^2}$

$$\int_0^3 \sqrt{9-x^2}\, dx = \int_{x=0}^{3} 3\cos\theta \, 3\cos\theta \, d\theta$$

$$= \int_{x=0}^{3} 9\cos^2\theta \, d\theta = 9\int_{x=0}^{3} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{9\theta}{2} + \frac{9\sin 2\theta}{4}\Big|_{x=0}^{3}$$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$ $\quad \sin 2\theta = 2\sin\theta\cos\theta = 2\,\frac{x}{3}\,\frac{\sqrt{9-x^2}}{3}$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{18}{12\cdot 9} \times \sqrt{9-x^2}\,\Big|_0^3$$

$$= \left(\frac{9}{2}\sin^{-1}(1) + 0\right) - \left(\frac{9}{2}\sin^{-1}(0) + 0\right)$$

$$= \frac{9}{2}\cdot\frac{\pi}{2} - \frac{9}{2}\cdot 0$$

$$= \boxed{\frac{9\pi}{4}}$$

(ii)


$\frac{1}{4}$ area of circle of radius 3

$$= \frac{1}{4}\pi(3)^2 = \boxed{\frac{9}{4}\pi}$$