15. (7pts) Polonium-210 (symbol Po, atomic number 84), the radioactive compound used recently to kill Russian dissident Alexander Litvinenko, has a half-life of 138 days.

(a) Write down the general solution for a radioactive decay equation, or re-derive it from the differential equation \( \frac{dy}{dt} = ky \), where \( t \) is in days. You do not have to show work on this part.

(b) Write down the particular solution corresponding to an initial amount of 2 (grams) at time \( t = 0 \).

(c) Determine the value of \( k \) in the governing differential equation \( \frac{dy}{dt} = ky \).

(d) Compute the time at which only 10% of the initial amount remains.

\[(a) \quad A(t) = A_0 e^{kt}
\]

\[(b) \quad A(0) = 2
\]

\[A_0 e^0 = 2
\]

\[A_0 = 2
\]

\[A(t) = 2 e^{kt}
\]

\[(c) \quad A(\frac{138}{2}) = 1
\]

\[2 e^{\frac{138}{2} k} = 1
\]

\[e^{\frac{138}{2} k} = \frac{1}{2}
\]

\[\frac{138}{2} k = \ln \frac{1}{2}
\]

\[k = \frac{\ln \frac{1}{2}}{138}
\]
10. (5 pts) Find the limit $\lim_{x \to 0^+} x \ln x$.

$$\lim_{x \to 0^+} x = 0 \quad \lim_{x \to 0^+} \ln x = -\infty \quad \text{IF type } 0(-\infty)$$

$$\lim_{k \to 0^+} k \ln k = \lim_{k \to 0^+} \frac{\ln k}{\frac{1}{k}} = \lim_{k \to 0^+} \frac{k^{1/k}}{\frac{1}{k}} = \lim_{k \to 0^+} -k = 0.$$

11. (5 pts) Compute the derivative of the function $f(x) = (3x - 2)^{\arcsin x}$.

$$y = (3x - 2)^{\arcsin x}$$

$$\ln y = \arcsin x \ln (3x - 2)$$

$$\frac{y'}{y} = \frac{1}{\sqrt{1-x^2}} \ln(3x-2) + \frac{3x}{3x-2} \frac{3x}{3x-2}$$

$$y' = \left( \frac{\ln(3x-2)}{\sqrt{1-x^2}} + \frac{3x}{3x-2} \right) (3x-2)^{\arcsin x}$$

12. (4 pts) Recall that by definition, $\ln x := \int_1^x \frac{1}{t} \, dt$. On the axes below, illustrate $\ln \left( \frac{1}{2} \right)$ as an area, and label all relevant quantities. Explain how we can understand from the illustration that $\ln \left( \frac{1}{2} \right)$ is negative.

[Diagram showing the area under the curve of $\frac{1}{t}$ from 1 to $\frac{1}{2}$, labeled as $\ln \frac{1}{2}$, and the line $y = \frac{1}{x}$, with an area highlighted and labeled as negative b/c backwards integration.]
6. (7pts) Compute the Taylor Series $T(x)$ centered at 0 for $f(x) = (1 + x)^{-3}$.

\[ f(x) = (1 + x)^{-3} \quad f(0) = 1 \]
\[ f'(x) = -3(1 + x)^{-4} \quad f'(0) = -3 \]
\[ f''(x) = 4.3(1 + x)^{-5} \quad f''(0) = 4.3 \]
\[ f'''(x) = -5.4.3(1 + x)^{-6} \quad f'''(0) = -5.4.3 \]
\[ f^{(n)}(x) = (-1)^n(n+2)(n+1)...(3)(1+x)^{-(n+3)} \quad f^{(n)}(0) = (-1)^n(n+2)...(3) \]

\[ f^{(n)}(x) = (-1)^n\frac{(n+2)!}{2} (1+x)^{-(n+3)} \quad f^{(n)}(0) = (-1)^n\frac{(n+2)!}{2} \]

\[ T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{\sum_{n=0}^{\infty} (-1)^n(n+2)!}{2} \frac{x^n}{n!} \]

7. (7pts) Compute the 4th degree Taylor Polynomial $T_4(x)$ centered at 1 for the function $f(x) = \ln x$.

\[ f(x) = \ln x \quad f(1) = 0 \]
\[ f'(x) = x^{-1} \quad f'(1) = 1 \]
\[ f''(x) = -x^{-2} \quad f''(1) = -1 \]
\[ f'''(x) = 2x^{-3} \quad f'''(1) = 2 \]
\[ f^{(4)}(x) = -6x^{-4} \quad f^{(4)}(1) = -6 \]

\[ T_4(x) = \sum_{i=0}^{4} \frac{f^{(i)}(1)}{i!} (x-1)^i \]

\[ = 1 \cdot (x-1) - \frac{1}{2} (x-1)^2 + \frac{1}{3} (x-1)^2 - \frac{1}{4} (x-1)^3 \]
3. (8pts) The plot below is generated by a particle following the parametric equations

\[
x(t) = 2\sin t \\
y(t) = e^t \cdot \cos t \\
0 \leq t \leq \pi.
\]

Determine exactly the maximum vertical position \(y(t)\) of the particle, and the value of \(t\) for which this occurs.

\[
\text{solve } y'(t) = 0 \text{ for } t \\
y'(t) = e^t (-\sin t) + e^t \cos t \\
0 = e^t (-\sin t + \cos t) \\
0 = -\sin t + \cos t \\
\sin t = \cos t \\
\text{unique solution in } [0, \pi] \\
is \frac{\pi}{4}.
\]

\[
y\left(\frac{\pi}{4}\right) = e^{\pi/4} \cos \frac{\pi}{4} = e^{\pi/4} \frac{\sqrt{2}}{2}
\]
1. (7pts) The slope field below is for the differential equation $y' = y - x$.
   
   (a) Approximate $y(1.5)$ assuming that $y(0) = \frac{1}{2}$ by sketching the particular solution on the slope field.
   
   (b) Approximate $y(1.5)$ assuming that $y(0) = \frac{1}{2}$ by using Euler's method with step size $\Delta x = 0.5$ (sketch Euler's method on the slope field).
   
   (Hint: In both cases, it is possible to check your answer exactly.)
2. (8pts) Sketch the plot of the polar function $r(\theta) = \sin(4\theta)$ on the range $0 \leq \theta \leq \frac{\pi}{2}$. 

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$4\theta$</th>
<th>$\sin(4\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\frac{\pi}{16}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{8}$</td>
<td>$\frac{\pi}{2}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\frac{3\pi}{16}$</td>
<td>$\frac{3\pi}{4}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$\pi$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\frac{5\pi}{8}$</td>
<td>$\frac{5\pi}{4}$</td>
<td>$-\frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>$\frac{7\pi}{16}$</td>
<td>$\frac{7\pi}{4}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$2\pi$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
4. (6pts) Use an appropriate test to determine the convergence of the series \( \sum_{n=1}^{\infty} n \cdot e^{-n^2} \).

The preconditions of the test must be mentioned individually but do not have to be checked.

\[
\int_{1}^{\infty} xe^{-x^2} \, dx = \left[ -\frac{e^{-u^2}}{2} \right]_{1}^{\infty} = -\frac{e^{-1}}{2} \\
\text{Series converges by int. test.}
\]

5. (6pts) Determine exactly the entire interval of convergence of the power series

\[
\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 \cdot 4^n}.
\]

**Ratio Test:**

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2 \cdot 4^{n+1}} \right| \cdot \frac{(x-2)^n}{n^2 \cdot 4^n} = \lim_{n \to \infty} \left| \frac{(x-2)}{n+1} \right| = \frac{|x-2|}{4} < 1
\]

- When \( |x-2| < 4 \)
  - \(-4 < x-2 < 4 \)
  - \(-2 < x < 6 \)

\( x = -2 \):

\[
\sum_{n=1}^{\infty} \frac{(-4)^n}{n^2 \cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{ converges by Alt. Con. Test}
\]

\( x = 6 \):

\[
\sum_{n=1}^{\infty} \frac{4^n}{n^2 \cdot 4^n} = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges, p-series, } p > 1.
\]

**Interval of convergence is** \([-2, 6]\).
8. (7pts) By converting the integrand into a power series, evaluate the indefinite integral

\[ \int x \cos(x^3) \, dx. \]

From from \( \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \) for all \(-\infty < x < \infty\),

\[ x \cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} \]

So \( \int x \cos(x^3) \, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{6n+1} \).

9. (7pts) Use Taylor's Inequality (i.e., the Taylor Remainder estimate) to find the number of terms of the Taylor Series \( T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) for \( f(x) = e^x \) needed to estimate \( e^{0.1} \) to within \( 10^{-5} \).

The estimate is \( T_n(0.1) = \sum_{i=0}^{n} \frac{(0.1)^i}{i!} \).

From from \( |R_n(x)| \leq \frac{M |x - 0|^{n+1}}{(n+1)!} \),

centered on \( 0 \), need \( 0.1 \), so \( d = 0.1 \).

\( f^{(n+1)}(x) = e^x \), so \( |f^{(n+1)}(x)| = e^x \leq e^{0.1} \) on \([-0.1, 0.1]\).

Take \( M = e^{0.1} \) since \( e^{0.1} < 3 \) clearly.

\[ |R_n(0.1)| \leq \frac{M |0.1|^{n+1}}{(n+1)!} \]

\( n=4 \) : \( \frac{3}{5!} (0.1)^5 < 10^{-5} \) clearly.

\( n=3 \) : \( \frac{3}{24} (0.1)^4 > 10^{-5} \).

So \( T_4(0.1) \) is the best choice for this \( M=3 \).
13. (8pts) Evaluate the integral \( \int_1^\infty x \cdot e^{-3x} \, dx \).

\[
\begin{align*}
\int_1^\infty x \cdot e^{-3x} \, dx &= \left[ -\frac{x}{3} e^{-3x} - \frac{1}{9} e^{-3x} \right]_1^\infty \\
&= 0 - \left( -\frac{1}{3} e^{-3} - \frac{1}{9} e^{-3} \right) \\
&= e^{-3} \left( \frac{1}{3} + \frac{1}{9} \right) \\
&= \frac{4}{9 e^3}
\end{align*}
\]

\[\therefore S = 6 + A\]

14. (8pts) Evaluate the integral \( \int \frac{1}{(1 + x^2)^2} \, dx \).

\[
\begin{align*}
&x = \tan \theta \\
&dx = \sec^2 \theta \, d\theta \\
&\sec \theta = \sqrt{1 + x^2} \\
&\sec^2 \theta = (1 + x^2)^2 \\
&\int \frac{1}{(1 + x^2)^2} \, dx = \int \frac{\sec^2 \theta \, d\theta}{\sec^2 \theta} = \int d\theta = \cos \theta + C \\
&= \int \frac{1 + \cos^2 \theta}{2} \, d\theta \\
&= \frac{\theta}{2} + \frac{\sin \theta}{4} + C \\
&\sec \theta = \sqrt{1 + x^2} \\
&\cos \theta = \frac{1}{\sqrt{1 + x^2}} \\
&\sin \theta = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} \\
&\sin \theta = \frac{2x}{\sqrt{1 + x^2} \sqrt{1 + 4x^2}} = \frac{2x}{1 + x^2}
\end{align*}
\]