15. **(7pts)** Polonium-210 (symbol Po, atomic number 84), the radioactive compound used recently to kill Russian dissident Alexander Litvinenko, has a half-life of 138 days.

(a) Write down the general solution for a radioactive decay equation, or re-derive it from the differential equation $\frac{dy}{dt} = ky$, where t is in days. You do not have to show work on this part.

(b) Write down the particular solution corresponding to an initial amount of 2 (grams) at time $t = 0$.

(c) Determine the value of k in the governing differential equation $\frac{dy}{dt} = ky$.

(d) Compute the time at which only 10% of the initial amount remains.

(a) $A(t) = A_0 e^{kt}$

(b) $A(0) = 2$

$A_0 e^0 = 2$

$A_0 = 2$

$A(t) = 2e^{kt}$

(c) $A(138) = 1$

$2e^{138k} = 1$

$e^{138k} = \frac{1}{2}$

$138k = \ln\frac{1}{2}$

$k = \frac{\ln\frac{1}{2}}{138}$

10. **(5pts)** Find the limit $\lim_{x \to 0+} x \cdot \ln x$.

$\lim_{x\to 0^+} x = 0$ $\quad \lim_{x\to 0^+} \ln x = -\infty$ $\quad$ IF type $0(-\infty)$

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

11. **(5pts)** Compute the derivative of the function $f(x) = (3x-2)^{\arcsin x}$.

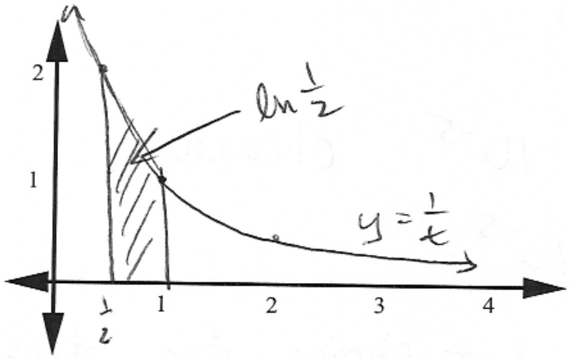
$$y = (3x-2)^{\sin^{-1}x}$$

$$\ln y = \sin^{-1}x \, \ln(3x-2)$$

$$\frac{y'}{y} = \frac{1}{\sqrt{1-x^2}} \ln(3x-2) + \sin^{-1}x \, \frac{3}{3x-2}$$

$$y' = \left(\frac{\ln(3x-2)}{\sqrt{1-x^2}} + \frac{3\sin^{-1}x}{3x-2}\right)(3x-2)^{\sin^{-1}x}$$

12. **(4pts)** Recall that by definition, $\ln x := \int_1^x \frac{1}{t}\, dt$. On the axes below, illustrate $\ln\left(\frac{1}{2}\right)$ as an area, and label all relevant quantities. **Explain** how we can understand from the illustration that $\ln\left(\frac{1}{2}\right)$ is negative.



negative b/c backwards integration

6. **(7pts)** Compute the Taylor Series $T(x)$ centered at 0 for $f(x) = (1+x)^{-3}$.

$f(x) = (1+x)^{-3}$ $\quad f(0) = 1$

$f'(x) = -3(1+x)^{-4}$ $\quad f'(0) = -3$

$f''(x) = 4\cdot 3(1+x)^{-5}$ $\quad f''(0) = 4\cdot 3$

$f'''(x) = -5\cdot 4\cdot 3(1+x)^{-6}$ $\quad f'''(0) = -5\cdot 4\cdot 3$

$\vdots$

$f^{(n)}(x) = (-1)^n (n+2)(n+1)\cdots(3)(1+x)^{-(n+3)}$ $\quad f^{(n)}(0) = (-1)^n (n+2)\cdots(3)$

$f^{(n)}(x) = (-1)^n \frac{(n+2)!}{2}(1+x)^{-(n+3)}$ $\quad f^{(n)}(0) = \frac{(-1)^n (n+2)!}{2}$

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)!}{2\cdot n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)}{2} x^n$$

7. **(7pts)** Compute the 4th degree Taylor Polynomial $T_4(x)$ centered at 1 for the function $f(x) = \ln x$.

$f(x) = \ln x$ $\quad f(1) = 0$

$f'(x) = x^{-1}$ $\quad f'(1) = 1$

$f''(x) = -x^{-2}$ $\quad f''(1) = -1$

$f'''(x) = 2x^{-3}$ $\quad f'''(1) = 2$

$f^{(4)}(x) = -6x^{-4}$ $\quad f^{(4)}(1) = -6$

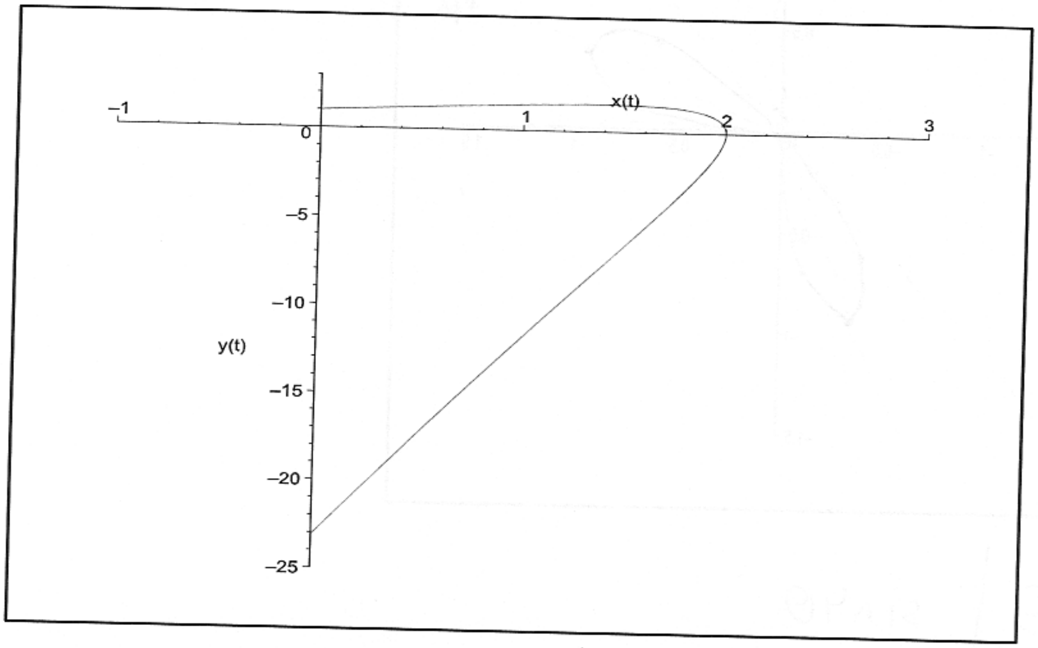
$$T_4(x) = \sum_{i=0}^{4} \frac{f^{(i)}(1)}{i!}(x-1)^i$$

$$= 1\cdot(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^2 - \frac{1}{4}(x-1)^3$$

3. **(8pts)** The plot below is generated by a particle following the parametric equations

$$\begin{aligned} x(t) &= 2\sin t \\ y(t) &= e^t \cdot \cos t \\ 0 &\le t \le \pi. \end{aligned}$$

Determine **exactly** the maximum vertical position $y(t)$ of the particle, and the value of t for which this occurs.



Solve $y'(t) = 0$ for t

$y'(t) = e^t(-\sin t) + e^t \cos t$

$0 = e^t(-\sin t + \cos t)$

$0 = -\sin t + \cos t$

$\sin t = \cos t$

unique solution in $[0, \pi]$

is $\frac{\pi}{4}$.

$y(\frac{\pi}{4}) = e^{\pi/4} \cos\frac{\pi}{4} = e^{\frac{\pi}{4}} \frac{\sqrt{2}}{2}$

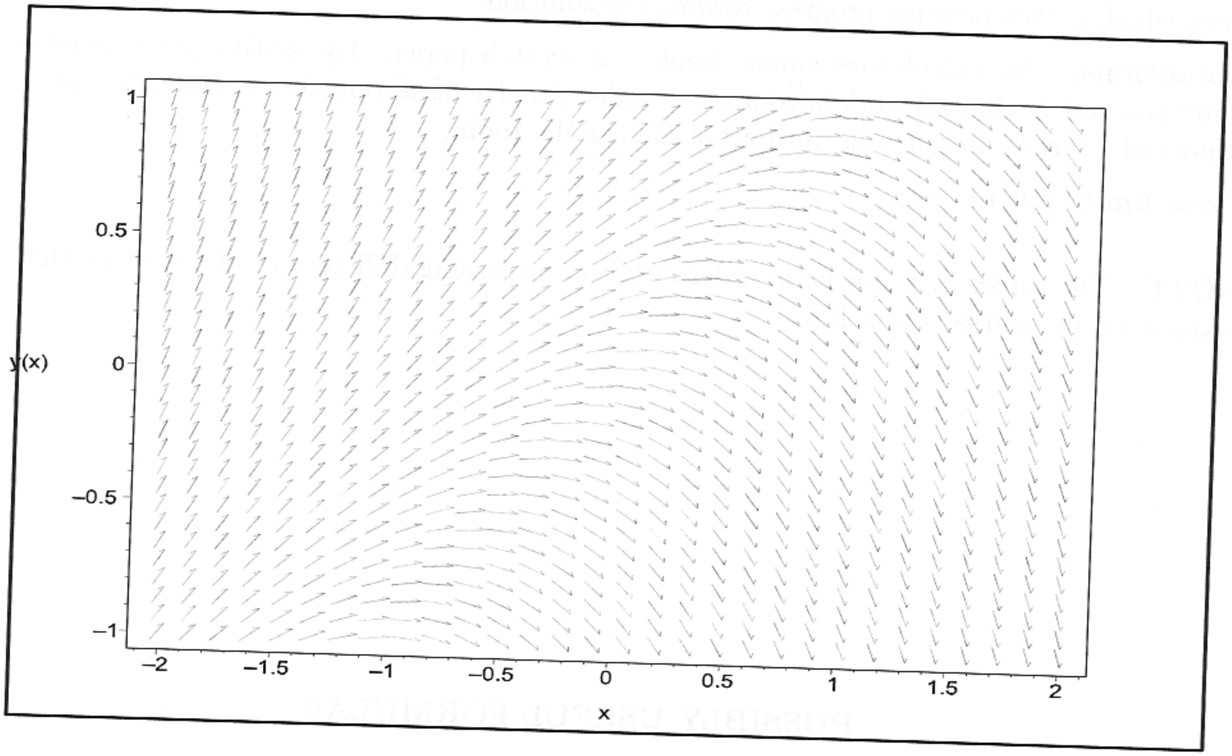
SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. **(7pts)** The slope field below is for the differential equation $y' = y - x$.

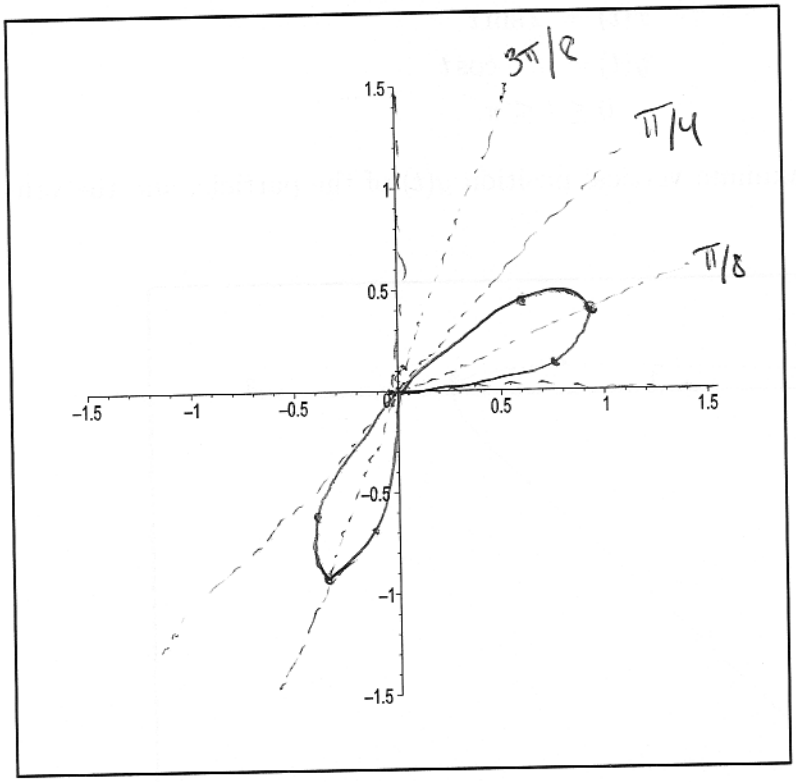
(a) Approximate $y(1.5)$ assuming that $y(0) = \frac{1}{2}$ by sketching the particular solution on the slope field.

(b) Approximate $y(1.5)$ assuming that $y(0) = \frac{1}{2}$ by using Euler's method with step size $\Delta x = 0.5$ (sketch Euler's method on the slope field).

(**Hint:** In both cases, it is possible to check your answer exactly.)



2. **(8pts)** Sketch the plot of the **polar** function $r(\theta) = \sin(4\theta)$ on the range $0 \le \theta \le \frac{\pi}{2}$.



θ	4θ	$\sin 4\theta$
0	0	0
$\frac{\pi}{16}$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{8}$	$\frac{\pi}{2}$	1
$\frac{3\pi}{16}$	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{4}$	π	0
$\frac{5\pi}{16}$	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{8}$	$\frac{3\pi}{2}$	-1
$\frac{7\pi}{16}$	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	2π	0

4. **(6pts)** Use an appropriate test to determine the convergence of the series $\sum_{n=1}^{\infty} n \cdot e^{-n^2}$.

The preconditions of the test must be mentioned individually but do not have to be checked.

$f(x) = xe^{-x^2}$ pos, decr., cont.

$$\int_1^\infty xe^{-x^2}\,dx = \int_{x=1}^{\infty} -\frac{e^u}{2}\,du = -\frac{e^u}{2}\Big|_{x=1}^{\infty} = -\frac{e^{-x^2}}{2}\Big|_1^\infty$$

$u = -x^2$

$du = -2x\,dx$

$$= 0 - \left(-\frac{e^{-1}}{2}\right) = \frac{1}{2e}$$

$\therefore$ series converges by int. test.

5. **(6pts)** Determine exactly the entire interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 \cdot 4^n}.$$

ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{(n+1)^2 4^{n+1}}}{\frac{(x-2)^n}{n^2 4^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-2)n^2}{(n+1)^2 4}\right|$

$= \left|\frac{x-2}{4}\right|$ $\quad \left|\frac{x-2}{4}\right| < 1$ when

$|x-2| < 4$

$-4 < x-2 < 4$

$-2 < x < 6$

$x = -2$: $\sum_{n=1}^{\infty} \frac{(-4)^n}{n^2 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by Alt ser. test

$x = 6$: $\sum_{n=1}^{\infty} \frac{4^n}{n^2 4^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges p-series, $p > 1$.

interval of convergence is $[-2, 6]$.

8. **(7pts)** By converting the integrand into a power series, evaluate the indefinite integral

$$\int x \cos\left(x^3\right) dx .$$

from cover, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all $-\infty < x < \infty$

$$x\cos(x^3) = x \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!}$$

so $\int x\cos(x^3)dx = \int\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!}\right)dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{6n+1}}{(2n)!} dx$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!(6n+2)}$$

9. **(7pts)** Use Taylor's Inequality (i.e., the Taylor Remainder estimate) to find the number of terms of the Taylor Series $T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $f(x) = e^x$ needed to estimate $e^{0.1}$ to within 10^{-5}.

The estimate is $T_n(0.1) = \sum_{i=0}^{n} \frac{(0.1)^i}{i!}$

from cover, $|R_n(x)| \le \frac{M}{(n+1)!}|x-0|^{n+1}$

center=0, need 0.1, so $d = 0.1$.

$f^{(n+1)}(x) = e^x$, so $|f^{(n+1)}(x)| = e^x \le e^{0.1}$ on $[-0.1, 0.1]$

take $M = 3$ since $e^{0.1} < 3$ clearly.

$$|R_n(0.1)| \le \frac{3}{(n+1)!}(0.1)^{n+1}$$

$n=4$: $\frac{3}{5!}(0.1)^5 < 10^{-5}$ clearly.

$n=3$: $\frac{3}{24}(0.1)^4 > 10^{-5}$.

so $T_4(0.1)$ is the best choice for this $M=3$.

13. **(8pts)** Evaluate the integral $\int_1^{\infty} x \cdot e^{-3x}\, dx$.

x	e^{-3x}
1	$-\frac{1}{3}e^{-3x}$
	$\frac{1}{9}e^{-3x}$

$$= -\frac{x}{3}e^{-3x} - \frac{1}{9}e^{-3x}\Big|_1^{\infty}$$

$$= 0 - \left(-\frac{1}{3}e^{-3} - \frac{1}{9}e^{-3}\right)$$

$$= e^{-3}\left(\frac{1}{3} + \frac{1}{9}\right)$$

$$= \frac{4}{9e^3}$$

14. **(8pts)** Evaluate the integral $\int \frac{1}{(1+x^2)^2}\, dx$.

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$\sec\theta = \sqrt{1+x^2}$

$\sec^4\theta = (1+x^2)^2$

$$\int \frac{1}{1+x^2}\, dx = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \frac{d\theta}{\sec^2\theta} = \int \cos^2\theta\, d\theta$$

$$= \int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$

$$= \frac{\tan^{-1}x}{2} + \frac{x}{2(1+x^2)} + C$$

$\theta = \tan^{-1}x \qquad \sin 2\theta = 2\sin\theta\cos\theta = \frac{2x}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+x^2}} = \frac{2x}{1+x^2}$