14. **(6pts)** Use an appropriate test to determine the convergence of the series $\sum_{n=3}^{\infty} \frac{\ln n}{n}$. The preconditions of the test must be mentioned individually but do not have to be checked.

Integral test. $f(x) = \frac{\ln x}{x}$ positive, decreasing, continuous

$$\int_3^\infty \frac{\ln x}{x}\,dx = \int_{x=3}^{\infty} u\,du = \frac{u^2}{2}\Big|_{x=3}^{\infty} = \frac{(\ln x)^2}{2}\Big|_3^\infty = \infty$$

$u = \ln x$

$du = \frac{dx}{x}$

$\therefore$ series diverges.

15. **(6pts)** Determine exactly the entire interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n^3(x-2)^n}{3^n}.$$

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \dfrac{\frac{(n+1)^3(x-2)^{n+1}}{3^{n+1}}}{\frac{n^3(x-2)^n}{3^n}} \right|$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^3 (x-2)}{n^3\ 3} \right| = \left|\frac{x-2}{3}\right|$$

$\left|\frac{x-2}{3}\right| < 1$ when $|x-2| < 3$

$-3 < x-2 < 3$

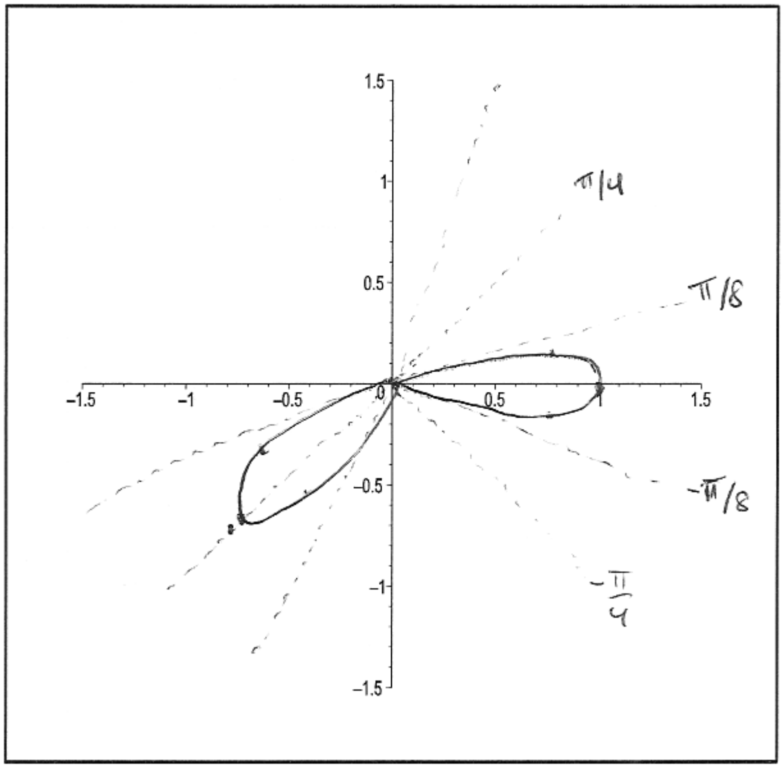
$-1 < x < 5$

$x = -1$: $\sum_{n=1}^{\infty} \frac{n^3(-3)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n n^3$ diverges by divergence test

$x = 5$: $\sum_{n=1}^{\infty} \frac{n^3 3^n}{3^n} = \sum_{n=1}^{\infty} n^3$ diverges by div. test

interval of conv: $(-1, 5)$.

12. **(8pts)** Sketch the plot of the **polar** function $r(\theta) = \cos(4\theta)$ on the range $-\frac{\pi}{8} \le \theta \le \frac{3\pi}{8}$.



	4θ	$\cos 4\theta$
$-\frac{\pi}{8}$	$-\frac{\pi}{2}$	0
$-\frac{\pi}{16}$	$-\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
0	0	1
$\frac{\pi}{16}$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{8}$	$\frac{\pi}{2}$	0
$\frac{3\pi}{16}$	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{\pi}{4}$	π	-1
$\frac{5\pi}{16}$	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{8}$	$\frac{3\pi}{2}$	0

10. **(7pts)** Polonium-210 (symbol Po, atomic number 84), the radioactive compound used recently to kill Russian dissident Alexander Litvinenko, has a half-life of 138 days.

(a) Write down the general solution for a radioactive decay equation, or re-derive it from the differential equation $\frac{dy}{dt} = ky$, where t is in days.

(b) Write down the particular solution corresponding to an initial amount of 3 (grams) at time $t = 0$.

(c) Determine the value of k in the governing differential equation $\frac{dy}{dt} = ky$.

(d) Compute the time at which only 40% of the initial amount remains.

(a) $A(t) = A_0 e^{kt}$

(b) $A(0) = 3$

$A_0 e^0 = 3$

$A_0 = 3$

so $A(t) = 3e^{kt}$

(c) $A(138) = 1.5$

$3e^{k138} = 1.5$

$e^{k \cdot 138} = \frac{1}{2}$

$k \cdot 138 = \ln\frac{1}{2}$

$k = \frac{\ln\frac{1}{2}}{138}$

(d) solve $A(t) = .4A_0$ for t

$A(t) = .4$

$3e^{\frac{\ln\frac{1}{2}}{138}t} = (.4)3$

$e^{\frac{\ln\frac{1}{2}}{138}t} = .4$

$\frac{\ln\frac{1}{2}}{138}t = \ln .4$

$t = 138 \frac{\ln .4}{\ln\frac{1}{2}}$

5. **(5pts)** Find the limit $\lim_{x \to 0^+} x \cdot \ln x$.

$\lim_{x\to 0^+} x = 0 \quad \lim_{x\to 0^+} \ln x = -\infty$

IF type $0(-\infty)$

$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$

6. **(5pts)** Compute the derivative of the function $f(x) = (1+2x)^{\arctan x}$.

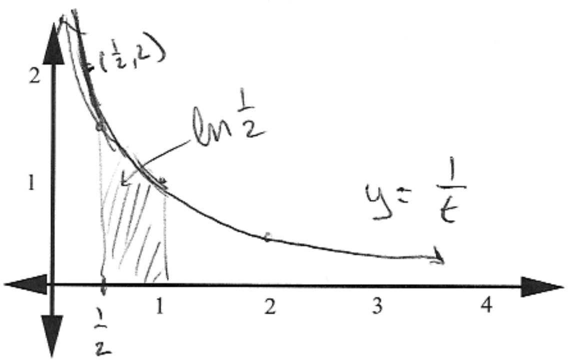
$y = (1+2x)^{\tan^{-1}x}$

$\ln y = \tan^{-1}x \ln(1+2x)$

$\frac{y'}{y} = \tan^{-1}x \frac{2}{1+2x} + \frac{1}{1+x^2}\ln(1+2x)$

$y' = \left(\frac{2\tan^{-1}x}{1+2x} + \frac{\ln(1+2x)}{1+x^2}\right)(1+2x)^{\tan^{-1}x}$

7. **(4pts)** Recall that by definition, $\ln x := \int_1^x \frac{1}{t}\,dt$. On the axes below, illustrate $\ln\left(\frac{1}{2}\right)$ as an area, and label all relevant quantities. **Explain** how we can understand from the illustration that $\ln\left(\frac{1}{2}\right)$ is negative.



negative because integration is backwards

SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. **(7pts)** Compute the 4th degree Taylor Polynomial $T_4(x)$ centered at 1 for the function $f(x) = \ln x$.

$f(x) = \ln x \quad f(1) = 0 \quad c_0 = 0$

$f'(x) = \frac{1}{x} \quad f'(1) = 1 \quad c_1 = 1$

$f''(x) = -\frac{1}{x^2} \quad f''(1) = -1 \quad c_2 = -\frac{1}{2}$

$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2 \quad c_3 = \frac{1}{3}$

$f^{(4)}(x) = \frac{-6}{x^4} \quad f^{(4)}(1) = -6 \quad c_4 = -\frac{1}{4}$

$$T_4(x) = \sum_{i=0}^{4} \frac{f^{(i)}(1)}{i!}(x-1)^i = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

2. **(7pts)** Compute the Taylor Series $T(x)$ centered at 3 for $f(x) = e^x$.

$f(x) = e^x \quad f(3) = e^3$

$f'(x) = e^x \quad f'(3) = e^3$

$f^{(n)}(x) = e^x \quad f^{(n)}(3) = e^3$

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

3. **(7pts)** By converting the integrand into a power series, evaluate the indefinite integral

$$\int x \cos\left(x^3\right) dx \,.$$

from cover, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ (true for $-\infty < x < \infty$)

so $x\cos(x^3) = x \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!}$

thus $\int x\cos(x^3)\,dx = \int \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!}\right) dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{6n+1}}{(2n)!}\,dx$

$= \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!\,(6n+2)}}$

4. **(7pts)** Use the 3rd degree Taylor Polynomial

$$T_3(x) \;=\; \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3$$

to estimate $\sin(35^o) = \sin(7\pi/36)$. Find an upper bound on the error in the approximation using Taylor's Inequality (the Taylor Remainder bound). Do not attempt to convert expressions into decimal form.

The estimate is $T_3\left(\frac{7\pi}{36}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\pi}{36} - \frac{1}{4}\left(\frac{\pi}{36}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{\pi}{36}\right)^3$

from cover, $|R_3(x)| \le \frac{M}{(\cancel{n+1})!}\left|x - \frac{\pi}{6}\right|^4$.

note $|f^{(4)}(x)| \le 1$ for $\left|x - \frac{\pi}{6}\right| \le \frac{\pi}{36}$.

take $d = \frac{\pi}{36}$, $M = 1$.

$\left|R_3\left(\frac{7\pi}{36}\right)\right| \le \frac{1}{4!}\left|\frac{7\pi}{36} - \frac{\pi}{6}\right|^4 = \boxed{\frac{1}{4!}\left(\frac{\pi}{36}\right)^4}$

8. **(8pts)** Evaluate the integral $\displaystyle\int_1^\infty x \cdot e^{-2x}\,dx$.

x	e^{-2x}
1	$-\frac{1}{2}e^{-2x}$
	$\frac{1}{4}e^{-2x}$

$$= -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\Big|_1^\infty$$

$$= 0 - \left(-\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2}\right)$$

$$= \frac{3}{4e^2}$$

9. **(8pts)** Evaluate the integral $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx$.

$x = \tan\theta \qquad \sec\theta = \sqrt{1+x^2}$

$dx = \sec^2\theta\,d\theta \qquad \sec^4\theta = (1+x^2)^2$

$$\int \frac{1}{(1+x^2)^2}dx = \int \frac{1}{\sec^4\theta}\sec^2\theta\,d\theta = \int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \int \cos^2\theta\,d\theta$$

$$= \int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C = \frac{\tan^{-1}x}{2} + \frac{1}{2}\frac{x}{1+x^2} + C$$

$\theta = \tan^{-1}x$

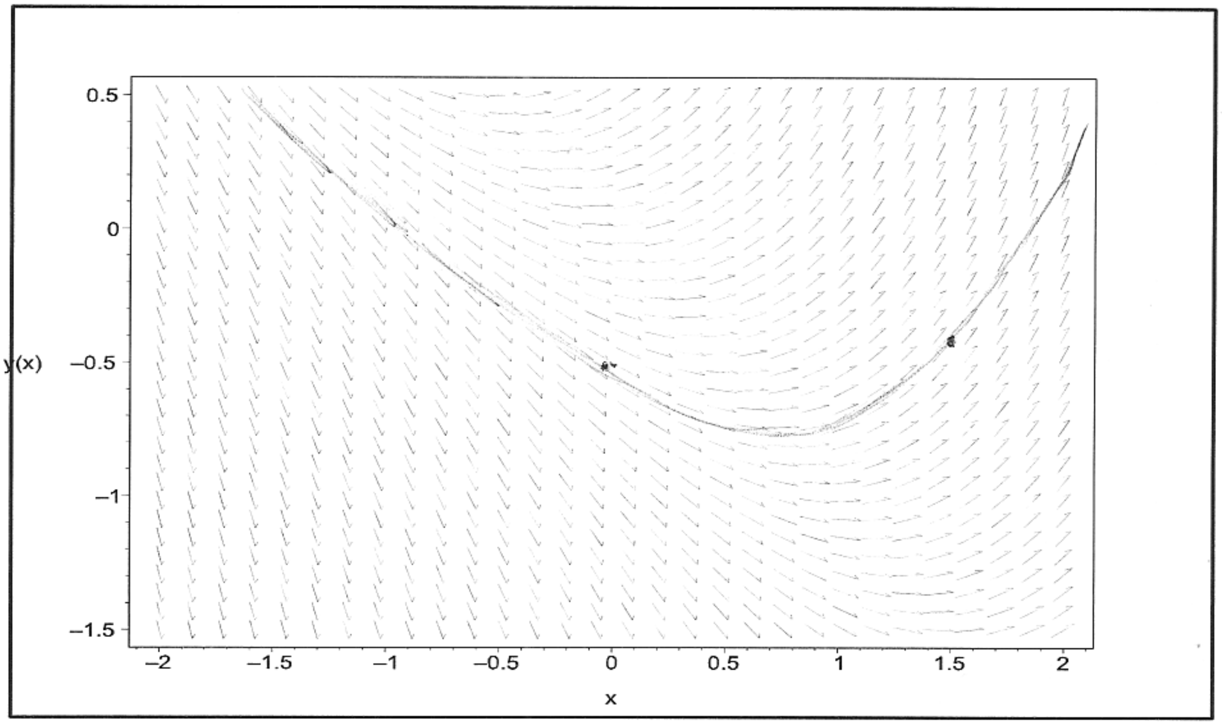
$\sin 2\theta = 2\sin\theta\cos\theta = 2\dfrac{x}{\sqrt{1+x^2}}\dfrac{1}{\sqrt{1+x^2}} = \dfrac{2x}{1+x^2}$

11. **(7pts)** The slope field below is for the differential equation $y' = x + y$.

(a) Approximate $y(1.5)$ assuming that $y(0) = -\frac{1}{2}$ by sketching the particular solution on the slope field.

(b) Approximate $y(1.5)$ assuming that $y(0) = -\frac{1}{2}$ by using Euler's method with step size $\Delta x = 0.5$ (sketch Euler's method on the slope field).

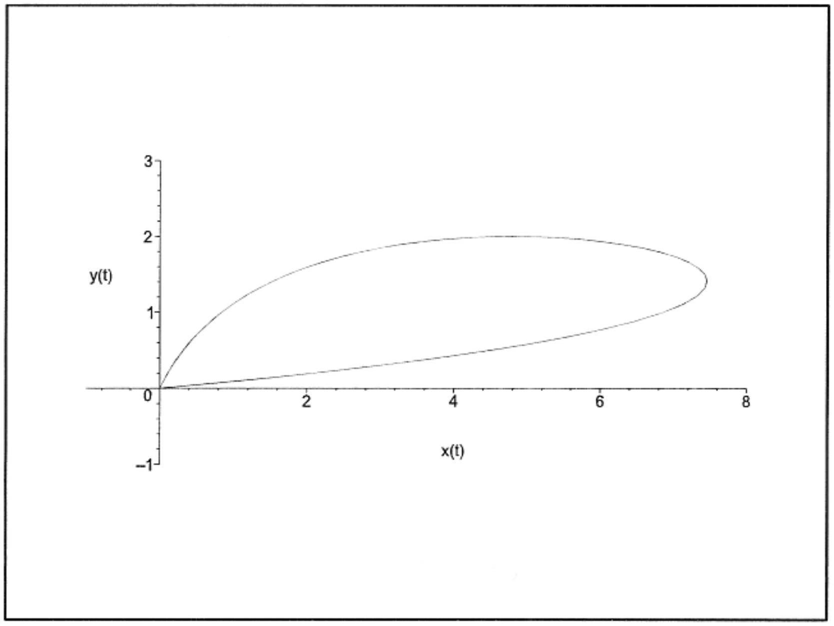
(**Hint:** In both cases, it is possible to check your answer exactly.)



13. **(8pts)** The plot below is generated by a particle following the parametric equations

$$\begin{aligned} x(t) &= e^t \cdot \sin t \\ y(t) &= 2\sin t \\ 0 \le t &\le \pi. \end{aligned}$$

Determine **exactly** the maximum horizontal position $x(t)$ of the particle, and the value of t for which this occurs.



solve $\frac{dx}{dt} = 0$ for t:

$$x'(t) = e^t \cos t + e^t \sin t$$
$$0 = e^t(\cos t + \sin t)$$
$$0 = \cos t + \sin t$$
$$\cos t = -\sin t$$

on $0 \le t \le \pi$ unique solution is

$$t = \frac{3\pi}{4}$$

$$x\left(\frac{3\pi}{4}\right) = e^{\frac{3\pi}{4}} \sin\frac{3\pi}{4} = e^{\frac{3\pi}{4}} \frac{\sqrt{2}}{2}$$