PRINT Last name: KEY First name: ____________

Signature: ____________ Student ID: ____________

Math 152 Exam 3, Fall 2005

Instructions. For the multiple choice problems, there is no partial credit. For the work-out problems, you must show the mathematical steps of the solution in order to receive full credit; writing only the answer will receive no credit. Written explanations for each step are not required (due to time constraint) but an incorrect solution with a correct written explanation might receive partial credit. You do not have to verify the preconditions of a convergence test unless it is explicitly requested.

Conditions. No calculators, computers, notes, books, or scratch paper. By writing your name on the exam you certify that all work is your own, under penalty of all remedies outlined in the IIT student rules. **Please do not talk until you are away from the room.**

Time limit: 50 minutes (strict).

NOTE: The topics may not be in order either of increasing difficulty or of the order they were covered in the course.

POSSIBLY USEFUL FORMULAS

$\sec^2 x = \tan^2 x + 1$

$M_n = \Delta x \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$

$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

$\int \tan x\, dx = -\ln|\cos x| + C$

$PV = nRT$

$|E_M| < \frac{K(b-a)^3}{24n^2} \quad (K \geq f''(x))$

$F = \rho g A d$

$|E_T| < \frac{K(b-a)^3}{12n^2} \quad (K \geq f''(x))$

$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$

$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$

$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$

$|E_S| < \frac{K(b-a)^5}{180n^4} \quad (K \geq f^{(4)}(x))$

$\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_n^{\infty} f(x)\, dx$

$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

$|s - s_n| \leq |a_{n+1}|$

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$\sin 2x = 2 \sin x \cos x$

$\text{Vol} = \int_a^b 2\pi\left[(f(x))^2 - (g(x))^2\right]\, dx$

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Part I. (10pts ea.) MULTIPLE CHOICE–NO PARTIAL CREDIT NO CALCULATORS

1. Compute the limit $\lim_{n\to\infty} \dfrac{\sqrt[4]{n^5-3n}}{\sqrt[4]{81n^5+4n^4-3n}}$.

a) $\frac{1}{27}$ b) $\frac{1}{9}$ c) $\boxed{\frac{1}{3}}$ d) $\frac{1}{4}$ e) 0

$$= \left(\lim_{n\to\infty} \frac{n^5-3n}{81n^5+4n^4-3n}\right)^{1/4}$$
$$= \left(\frac{1}{81}\right)^{1/4} \quad \text{since the leading powers of } n \text{ are both 5}$$
$$= \frac{1}{3}$$

2. Which statement is true about the series $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$?

a) It is a convergent p-series whose n^{th} partial sum is $s_n = -\ln(n+1)$.

$\boxed{\text{b)}}$ It is a divergent telescoping series whose n^{th} partial sum is $s_n = -\ln(n+1)$.

c) It is convergent because $\lim_{n\to\infty} \ln(n/(n+1)) = 0$.

d) It is divergent because $\lim_{n\to\infty} n/(n+1) = 1$.

e) It is convergent because $\ln(n/(n+1)) < 1/n$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent.

a) It is not a p-series. c) The nth term test does not give convergence, only divergence if the limit is nonzero. d) This limit is in fact 1, but this is not the limit of the nth term. e) The comparison is invalid – $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. To see that b) is true, note that

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots + (\ln n - \ln(n+1)) + \cdots$$

Since $\ln 1 = 0$, the nth term is $-\ln(n+1)$, which goes to $-\infty$.

3. The sum of the series $\frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \cdots$ is

a) ∞ b) 3 c) 2 $\boxed{\text{d)}}$ 1 e) $\frac{1}{3}$

The first term is $2/3$. The ratio is $(2/9)/(2/3) = 1/3$. The formula for geometric series gives

$$\frac{2/3}{1-1/3} = 1.$$

4. Select the correct completion of the statement: The series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4 - \frac{1}{2}}$

   a) converges by the ratio test.

   b) diverges by the ratio test.

   c) converges, by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

   $\boxed{\text{d)}}$ diverges, by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

   e) converges, by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$.

   a) and b) are false because the ratio test results in a limit of 1 which is inconclusive. c) is false because $1/n^3$ is not greater than $n^3/(n^4 - 1/2)$. e) is false because $1/(2n)$ is not less than $n^3/(n^4 - 1/2)$. d) is true since

$$\begin{aligned} \frac{1}{n} &\leq? \frac{n^3}{n^4 - 1/2} \\ n^4 - 1/2 &\leq? n^4, \end{aligned}$$

   which is true for all n. ($\leq?$ means we aren't sure if the inequality is true initially, but going backwards verifies it.)

5. Select the expression which represents the area bounded by one loop of the graph of the polar function $r = \sin(3\theta)$.

   a) $\displaystyle\int_{-\pi/6}^{\pi/6} \pi \sin^2(3\theta)\, d\theta$

   b) $\displaystyle\int_{-\pi/6}^{\pi/6} \sin(3\theta)\, d\theta$

   c) $\displaystyle\int_{0}^{2\pi/3} \sin(3\theta)\, d\theta$

   d) $\displaystyle\int_{0}^{2\pi/3} \frac{1}{2} \sin^2(3\theta)\, d\theta$

   $\boxed{\text{e)}}$ $\displaystyle\int_{0}^{\pi/3} \frac{1}{2} \sin^2(3\theta)\, d\theta$

   The formula for polar area is $\int_\alpha^\beta \frac{1}{2}(f(\theta))^2\, d\theta$. Answer d) is impossible since it corresponds to two loops of the graph: at all three values $\theta = 0, \pi/3, 2\pi/3$, $\sin(3\theta) = 0$. Answer e) is the only possible choice.

Part II. SHOW WORK FOR FULL CREDIT NO CALCULATORS

6. **(15 pts)** Use one of the integral remainder estimates to estimate the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ to within 2×10^{-4}. **Clearly identify** the estimate, error, and error bound. Show your work which guarantees that the estimate is within this amount of the actual sum.

	actual	estimate	error	error bound	tolerance
Way 1	s	s_n	$s - s_n$	$s - s_n \le \int_n^\infty f(x)\,dx$	2×10^{-4}
Way 2	s	$s_n + \frac{\int_n^\infty f(x)dx + \int_{n+1}^\infty f(x)dx}{2}$	$s -$ est.	$\lvert s - \text{est.}\rvert \le \frac{a_n}{2}$	2×10^{-4}

Only one way is required for the solution. First way 1. Plugging in $f(x) = \frac{1}{x^2}$, set RHS of error bound $\le$ tolerance and solve for n.

$$\begin{aligned} \int_n^\infty \frac{dx}{x^2} &\le \frac{2}{10^4} \\ \left.\frac{-1}{x}\right|_n^\infty = \frac{1}{n} &\le \frac{2}{10^4} \\ n &\ge \frac{10^4}{2} = 5000. \end{aligned}$$

Therefore s_{5000} is within 2×10^{-4} of s.

Now Way 2, again with $f(x) = \frac{1}{x^2}$.

$$\begin{aligned} \frac{a_n}{2} = \frac{1}{2n^2} &\le \frac{2}{10^4} \\ n^2 &\ge \frac{10^4}{4} \\ n &\ge \frac{100}{2} = 50. \end{aligned}$$

The estimate is

$$\begin{aligned} s_{50} + \frac{\int_{50}^\infty f(x)dx + \int_{51}^\infty f(x)dx}{2} &= s_{50} + \frac{\left.\frac{-1}{x}\right|_{50}^\infty + \left.\frac{-1}{x}\right|_{51}^\infty}{2} \\ &= s_{50} + \frac{\frac{1}{50} + \frac{1}{51}}{2}, \end{aligned}$$

which is within 2×10^{-4} of s.

7. **(15 pts)** Assume that $f(x) = \sin x$ has power series representation

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

with radius of convergence $R = \infty$. Find a power series representation for $g(x) = \cos x$ and determine, with justification, its radius of convergence.

Way 1. Just take the derivative of both sides. When we take the derivative of a power series, the radius of convergence stays the same.

$$\begin{aligned} \frac{d}{dx} \sin x &= \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ \cos x &= \sum_{n=0}^{\infty} \frac{d}{dx} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!}, \end{aligned}$$

with radius of convergence $R = \infty$ as described above.

Way 2. Note the identity $\cos x = \sin(x + \pi/2)$. So we can just replace x with $x + \pi/2$ in the power series for $\sin x$.

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x+\pi/2)^{2n+1}}{(2n+1)!},$$

which converges everywhere ($R = \infty$) since the original converged everywhere and we only plugged in a shifted value.

8. **(20 pts)** Find the center, radius and the **interval** of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x-3)^n}{n \cdot 5^n}.$$

The center of the power series is $\boxed{a = 3}$ from the way the series is presented.

Ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{(n+1)\cdot 5^{n+1}}}{\frac{(x-3)^n}{n\cdot 5^n}} \right| = \lim_{n\to\infty} \left| \frac{(x-3)n}{(n+1)5} \right| = \frac{|x-3|}{5},$$

and the ratio test says the series converges when this quantity is < 1 and diverges when > 1:

$$\begin{aligned} \frac{|x-3|}{5} &< 1 \\ |x-3| &< 5 \\ -5 < x-3 &< 5 \\ -2 < x &< 8. \end{aligned}$$

The radius of convergence is half the distance between the endpoints of the interval of convergence. Another way to see that $\boxed{R = 5}$ it is that $(-2, 8) = (3 - 5, 3 + 5)$. Finally we must check for convergence at the endpoints.

At $x = -2$:

$$\sum_{n=1}^{\infty}(-1)^n \frac{(-5)^n}{n \cdot 5^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges as a p-series with $p = 1$. (Or do integral test.)

At $x = 8$:

$$\sum_{n=1}^{\infty}(-1)^n \frac{(5)^n}{n \cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges because it is an alternating series.

Therefore the interval of convergence is $\boxed{(-2, 8]}$.