Density of normal binary covering codes

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Covering codes

**Setting** Hypercube $Q_n \equiv F_2^m \equiv \{0, 1\}^n$

Hamming distance $d(u, v) = \text{weight}(u - v)$

A code is a set $C \subseteq Q_n$ (n is the length of $C$)

The covering radius of $C$ is the smallest $R \geq 0$ such that $d(x, C) \leq R$ for all $x \in Q_n$.

The ball of radius $R$ centered at $u$ is $B_n(u, R) = \{v \in Q_n : d(u, v) \leq R\}$.

$R$ is the minimum such that $\bigcup_{c \in C} B_n(c, R) = Q_n$

A $(n, R)$-code $C$ is a (symmetric) covering code of length $n$ and covering radius $R$. The smallest size of such a code is defined as $K(n, R)$. 
Some questions for covering codes

Minimum size $K(n, R)$ of a code for given $n$, $R$?

Minimum radius $R$ of a size $K$ length $n$ code?

Explicit constructions for infinite families of codes?

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The sphere bound

Any $R$-ball in $Q_n$ has size $b_n(R) := \binom{n}{\leq R}$; thus

$$K(n, R) \geq \frac{2^n}{b_n(R)}.$$ 

The optimal density of a $(n, R)$-code is

$$\mu(n, R) := K(n, R) \frac{b_n(R)}{2^n}.$$ 

Main question what is asymptotic worst density

$$\mu^*(R) := \limsup_{n \to \infty} \mu(n, R)$$
Some previous results

(Conjecture) for constant $R$, $\mu^*(R) = 1$

(Cohen, Lobstein, Sloane ‘86) for constant $R$,
$\mu^*(R) < \frac{2^R R^R (R+1)}{R!}$

(Kabatyanskii, Panchenko, ‘88) $\mu^*(1) = 1$

(Krivelevich, Sudakov, Vu ‘03) for constant $R \geq 3$,
$\mu^*(R) \leq e(R \ln R + \ln R + \ln \ln R + 2)$

This talk Extension of KSV to normal codes

Define $K_\nu(n, R)$ to be the minimum size normal $(n, R)$-code. Define $\nu(n, R) = K_\nu(n, R) \frac{b_n(R)}{2^n}$, and

$\nu^*(R) := \lim \sup_{n \to \infty} \nu(n, R)$.

(E., Exoo) for constant $R \geq 3$,

$\nu^*(R) \leq e(R \ln R + \ln R + \ln \ln R + 2 + \ln 2)$. 

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Normal codes: what and why

For a coordinate \( i \in [n] \) and a code \( C \) define
\[
C_{0}^{(i)} := \{ c \in C : c_i = 0 \}, \quad C_{1}^{(i)} := \{ c \in C : c_i = 1 \}
\]

A \((n, R)\)-code is normal if \( \exists i \in [n] \) such that
\[
d(v, C_{0}^{(i)}) + d(v, C_{1}^{(i)}) \leq 2R + 1, \quad \forall v \in Q_n
\]

Direct sum of codes
Given a \((n_{A}, R_{A})\)-code \( A \) and a \((n_{B}, R_{B})\)-code \( B \),
\[
A \oplus B := \{(a, b) : a \in A, b \in B\}
\]
is a \((n_{A} + n_{B}, R_{A} + R_{B})\)-code of size \(|A||B|\).

Amalgamated direct sum of normal codes (Graham, Sloane ‘85) If in addition \( A \) and \( B \) are normal,
\[
A \hat{\oplus} B := \{(a, 0, b) : a0 \in A_{0}^{(n_{A})}, 0b \in B_{0}^{(1)}\}
\]
\[
\cup \{(a, 1, b) : a1 \in A_{1}^{(n_{A})}, 1b \in B_{1}^{(1)}\}
\]
is a \((n_{A} + n_{B} - 1, R_{A} + R_{B})\)-code of size \(|A||B|/2\).
Probabilistic deletion method bound

(Cooper, E., Kahng '02)

1. Choose $\Theta\left(\frac{2^n}{b(n,R)}\right)$ words randomly into set $S$

2. Let $T$ be the set of vertices left uncovered

$S \cup T$ is a $(n, R)$-code ($T$ is the patch for $S$).

Result $\mu^*(R) = O(\ln n)$.

(Improved by inductive application)
Normal code inductive density bound

**Target** a small normal \((n, R)\)-code.

Let \(n = n_1' + n_1 - 1\), \(R = R_1 + R_1'\).

\[Q_{n_1'} 0 \quad Q_{n_1'} 1\]

“almost” \((n_1', R_1')\)-code \(S\)

size \(K_\nu(n_1, R_1) \) \((n_1, R_1)\)-code \(B\)

“patch” \(T\)

size \(K_\nu(n_1, R) \) \((n_1, R)\)-code \(B'\)

Amalgamated semi-direct sum \((S \oplus B) \cup (T \oplus B')\) of size

\[
\sim \frac{x 2^{n_1'}}{2 b_{n_1'-1}(R_1')} K_\nu(n_1, R_1) + \frac{1}{2} 2^{n_1'+1} e^{-x} K_\nu(n_1, R)
\]

(KSV ‘03) optimization of \(n_1, R_1\) and analysis as \(n \to \infty\)
Normal asymmetric codes

Asymmetry restriction Balls centered at \( c \in C \) only cover vertices by changing 1's to 0's

Asymmetric sphere bound The typical ball is of size \( \left( \frac{n}{2} \right)^{\leq R} \), so optimal code size \( K^+(n, R) \) is roughly

\[
K^+(n, R) \geq \frac{2^n}{\left( \frac{n}{2} \right)^{\leq R}}.
\]

An asymmetric \((n, R)\)-code \( C \) is normal if there exists an \( i \in [n] \) such that \( \forall v \in Q_n \)

\[
2R + 1 \geq \begin{cases} 
    d(v, C_0^{(i)}) + d(v, C_1^{(i)}), & \text{if } v_i = 0 \\
    2d(v, C_1^{(i)}) + 1, & \text{if } v_i = 1.
\end{cases}
\]

\( K^+_\nu(n, R) \) is the optimal size of an asymmetric normal \((n, R)^+\)-code

\[
\nu_+(n, R) := K^+_\nu(n, R) \frac{\left( \frac{n}{2} \right)^{\leq R}}{2^n} \\
\nu^*_+(R) := \limsup_{n \to \infty} \nu_+(n, R)
\]
Best known bounds for $K^+(n, R)$

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Sources
Cooper, E., Kahng, ‘02
Applegate, Rains, Sloane ‘03
E., Exoo ‘03