Multichannel liar games with a fixed number of lies

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Outline

1. Background
   - Definition of liar games
   - Liar game sphere bound
2. The multiple channel extension
   - Lying over an undetermined channel
   - The multichannel sphere bound
3. Results for the multichannel liar game

Basic liar game setting

Two-person game:

1. Carole picks a number $x \in [n]$
2. Paul asks q questions to determine x.

Playing optimally, Carole employs an adversarial strategy; it's a perfect information game.

Catch: Carole is allowed to lie at most k times.

Paul's t-ary questions

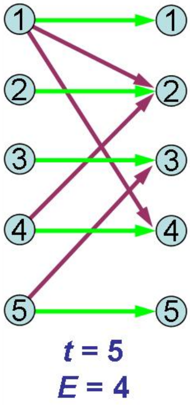
$t = 2$ (Binary coding).

- Paul selects a subset A of $[n]$ and asks "is $x \in A$?"
- Carole answers "Yes" or "No"

$t \geq 2$ (Non-binary coding).

- Paul t-partitions $[n] = A_1 \dot{\cup} \cdots \dot{\cup} A_t$ and asks for i such that $x \in A_i$
- Carole answers from $1, \ldots, t$

Restricting Carole's ability to lie



Select a communication channel C for the game.

1. Carole may always tell the truth.
2. Carole may use the E lie patterns

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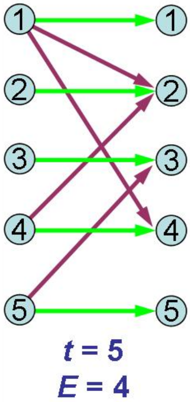

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Example round.

1. Paul asks $[10] = \{1,2\}\dot\cup\{3,4\}\dot\cup\{5,6\}\dot\cup\{7,8\}\dot\cup\{9,10\}$
2. Carole answers "3"
3. $\{5,6\}$ accumulate 0 lies
   $\{9,10\}$ accumulate 1 lie
   $\{1,2,3,4,7,8\}$ are eliminated

Liar game summary

Parameters:

- n, the number of possibilities
- q, the number of questions
- k, the (fixed) number of lies
- t, for t-ary questions
- E, the number of lie patterns in the channel

Given q, k, t, and E, after q rounds,

(Original) liar game: maximum n such that at most one element has $\leq k$ lies?
Pathological liar game: minimum n such that at least one element has $\leq k$ lies?

Estimating a sphere bound

q t-ary questions $\leftrightarrow$ t^q possible response sequences by Carole

Question. How many response sequences correspond to a typical element?

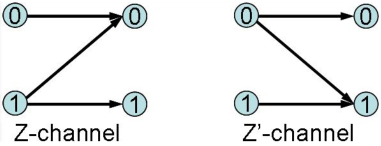
- E^k length k sequences of lie patterns
- $\binom{q}{k}$ positions for the lies
- $\sim \left(\frac{1}{t}\right)^k$ chance the positions are compatible with the lies

Total: $\sim \dfrac{E^k \binom{q}{k}}{t^k}$ response sequences per element

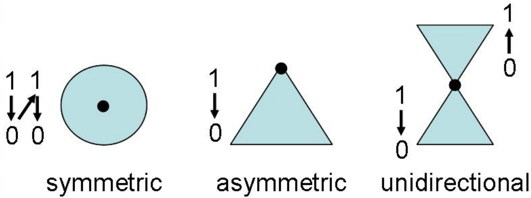
Expectation: Paul can win the liar game iff $n \lesssim \dfrac{t^{q+k}}{E^k \binom{q}{k}}$ (Proved by D-S).

The wrinkle – multiple channels

Question (N. Linial). Does it matter if Paul knows Carole is using a Z-channel but not which one?

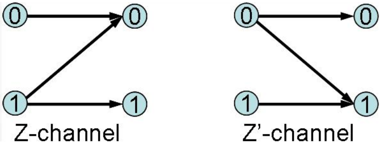


Equiv. Question. What is the liar game analog of packing with unidirectional Hamming balls?



The wrinkle – multiple channels

Question (N. Linial). Does it matter if Paul knows Carole is using a Z-channel but not which one? Answer: **Yes!**



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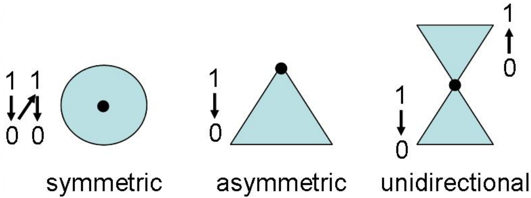

Liar games over an undetermined channel

Fix n, q, k, t, and $\mathcal{C} = \{C \in \mathcal{C}\}$ a set of t-ary channels.

Paul loses iff after q rounds,

- (Original) liar game: there are ≥ 2 pairs (x, C) where x has accumulated $\leq k$ lies over channel C.
- Pathological liar game: for every pair (x, C), x has accumulated $> k$ lies over channel C.

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- Pathological liar game: for every pair (x, C), x has accumulated $> k$ lies over channel C.

At the end of the original (pathological) liar game, Carole (Paul) selects the channel $C \in \mathcal{C}$ they were playing over!

Lie patterns for a multichannel

- A sequence of lie patterns is compatible with a channel if all lie patterns appear in the channel C.
- By $E\left(\cap_{C\in\mathcal{C}'}C\right)$ we mean the number of lie patterns common to all $C\in\mathcal{C}'$

Definition

Given k and a set $\mathcal{C}$ of t-ary channels,

$$E_k(\mathcal{C}) := \sum_{\mathcal{C}'\subseteq\mathcal{C}}(-1)^{|\mathcal{C}'|-1}E(\cap_{C\in\mathcal{C}'}C)^k.$$

is the number of length k sequence of lie patterns compatible with $\mathcal{C}$.

Estimating the multichannel sphere bound

- $E_k(\mathcal{C})$ length k sequences of lie patterns
- $\binom{q}{k}$ positions for the lies
- $\sim \left(\frac{1}{t}\right)^k$ chance the positions are compatible with the lies

Total: $\sim \dfrac{E_k(\mathcal{C})\binom{q}{k}}{t^k}$ response sequences per element

Expectation: Paul can win the

- (original) multichannel liar game iff $n \lesssim \dfrac{t^{q+k}}{E_k(\mathcal{C})\binom{q}{k}}$
- pathological multichannel liar game iff $n \gtrsim \dfrac{t^{q+k}}{E_k(\mathcal{C})\binom{q}{k}}$

Reducing adaptivity of Paul's questions

Remark. Usually, Paul adaptively uses the answers to all previous questions to determine his next question.

Batch questions. We make it harder for Paul by forcing him to put his q questions into two nonadaptive batches.

Here, batches have sizes $\sim (q - k \ln q)$ and $k \ln q$.

Main theorem

Theorem (E.-Nyman)

For all $\epsilon > 0$, for q large enough in the original multichannel liar game,

$$\begin{cases} \text{if } n \leq (1-\epsilon)\dfrac{t^{q+k}}{E_k(\mathcal{C})\binom{q}{k}}, & \text{then Paul wins;} \\ \text{if } n \geq (1+\epsilon)\dfrac{t^{q+k}}{E_k(\mathcal{C})\binom{q}{k}}, & \text{then Carole wins.} \end{cases}$$

For all $\epsilon > 0$, for q large enough in the pathological multichannel liar game,

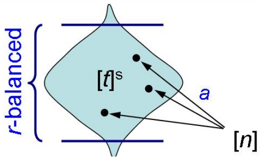
$$\left\{ \text{if } n \geq (1+\epsilon)\dfrac{t^{q+k}}{E_k(\mathcal{C})\binom{q}{k}}, \quad \text{then Paul wins.} \right.$$

Furthermore, in both games Paul's winning strategy uses two nonadaptive batches of questions, of sizes $\sim (q - k\ln q)$ and $k\ln q$, respectively.

Original liar game – first batch of questions (I)

Remark. Tools are adapted from Dumitriu-Spencer.

Definition. A string $w \in [t]^s$ is r-balanced if for all $i \in [t]$, i appears at most $s/t + r$ times in w.



- Find $s \sim (q - k \ln q)$, $a \in \mathbb{N}$, $r = s^{2/3}$
- $a \doteq (1 - \epsilon)\frac{t^{q+k-s}}{E_k(\mathcal{C})\binom{q}{k}}$
- identify a elements of $[n]$ with each r-balanced string

First batch: Paul asks for bits $1, \ldots, s$ of Carole's element

Original liar game – first batch of questions (II)

Definition. Let x_j be the number of elements with j lies assigned, giving state vector $(x_0, \ldots, x_k)$.

Observation. Carole must answer with a nearly r-balanced string w.

Lemma. For $0 \leq j \leq k$, $x_j \leq a\frac{E_j(\mathcal{C})}{j!}\left(\frac{s}{t} + r + k\right)^j$.

To work backwards from w to possible w' with j lies:

- $E_j(\mathcal{C})$ length j sequences of lie patterns
- $\leq \left(\frac{s}{t} + r + k\right)^j$ ways to apply each pattern to w
- Overcounted by $j!$
- Finally, a elements per w'.

Corollary. $x_k \leq (1 - \epsilon')t^{q-s}$

Original liar game – second batch of questions

Lemma. If there is a packing of x_j radius $(k-j)$ Hamming balls in $[t]^{q-s}$, $0 \leq j \leq k$, then Paul can win from state $(x_0, \ldots, x_k)$ in $q-s$ questions.

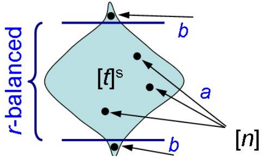
Proof sketch.

1. Identify element counted by x_j with center of a radius $(k-j)$ Hamming ball
2. Paul asks for bits $1, \ldots, q-s$ of the element
3. The packing condition guarantees only one element survives

Greedy packing. Doubling the radii of all Hamming balls, there are fewer than t^{q-s} vertices. (Dominating term is $x_k \leq (1-\epsilon')t^{q-s}$.) □

Pathological liar game – first batch of questions (I)

Definition. A string $w \in [t]^s$ is r-balanced if for all $i \in [t]$, i appears at most $s/t + r$ times in w.



- Find $s \sim (q - k \ln q)$, $a \in \mathbb{N}$, $r = s^{2/3}$
- $a \doteq (1+\epsilon)\frac{t^{q+k-s}}{E_k(\mathcal{C})\binom{q}{k}}$, $b \doteq (1+\epsilon)\frac{t^{q-s+k} k \ln(q-s)}{E_k(\mathcal{C})\binom{q-s}{k}}$
- identify a elements of $[n]$ with each r-balanced string, put b elements at extremes

First batch: Paul asks for bits $1, \ldots, s$ of Carole's element

Pathological liar game – first batch of questions (II)

Carole answers $w \in [t]^s$.

Case 1: w r-balanced. Then

$$x_k \geq a\frac{E_k(\mathcal{C})}{k!}\left(\frac{s}{t}-(t-1)r-k\right)^k \geq (1+\epsilon')t^{q-s}.$$

Case 2: w not r-balanced. Then $x_0 = b = (1+\epsilon)\frac{t^{q-s+k}}{E_k(\mathcal{C})}\ln(q-s)^k$.

Definition. A j-shadow of w is all w' obtained by applying to w a length $\leq j$ sequence of lie patterns compatible with $\mathcal{C}$.

Remark. The typical size of a k-shadow in $[t]^{q-s}$ is $\sim \frac{E_k(\mathcal{C})\binom{q-s}{k}}{t^k}$.

Pathological liar game – second batch of questions

Lemma. If there is a covering of $[t]^{q-s}$ with x_j $(k-j)$-shadows, then Paul can win from state $(x_0, \ldots, x_k)$ in $q-s$ questions.

- Case 1: Cover $[t]^{q-s}$ with $x_k > t^{q-s}$ 0-shadows
- Case 2: Cover $[t]^{q-s}$ greedily with $x_0 = b = (1+\epsilon)\frac{t^{q-s+k}}{E_k(\mathcal{C})\binom{q-s}{k}} \cdot \ln(q-s)^k$ k-shadows.

(Greedy covering of "middle" of $[t]^{q-s}$ based on minimum degree of k-shadow.) □

Concluding remarks and open questions

- Can we get the second term asymptotics for arbitrary $\mathcal{C}$?
- Modifications, such as bounding the number of times a lie pattern can be used, or imposing costs.
- Can we further reduce or eliminate completely the adaptiveness?
- Can these techniques be used to find the asymptotic best packings and coverings of $[t]^q$ with fixed-radius Hamming balls (unknown for radius ≥ 2)?
- Will these techniques work for coin-weighing, fault-testing, and related search problems?