A Kruskal-Katona Theorem for Cubical Complexes

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(ABSTRACT)

The optimal number of faces in cubical complexes which lie in cubes refers to the maximum number of faces that can be constructed from a certain number of faces of lower dimension, or the minimum number of faces necessary to construct a certain number of faces of higher dimension. If m is the number of faces of dimension r in such a cubical complex, and if $s > r(s < r)$, then the maximum(minimum) number of faces of dimension s that the complex can have is $m_{(s/r)} + (m - m_{(r/r)})^{(s/r)}$, in terms of upper and lower semipowers. The corresponding formula for simplicial complexes, proved independently by J. B. Kruskal and G. A. Katona, is $m^{(s/r)}$. A proof of the formula for cubical complexes is given in this paper, of which a flawed version appears in a paper by Bernt Lindström. The n-tuples which satisfy the optimality conditions for cubical complexes which lie in cubes correspond bijectively to f-vectors of cubical complexes.

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Finally, I am grateful to Bernt Lindström for his previous work on this subject. The main result is an extension of the formulas and theorems that he developed.

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1 Introduction

Some popular puzzles involve creating the maximum number of triangles or squares out of a given number of edges. For instance, how many squares could one construct using nine edges? The answer, three, is seen readily enough if the third dimension is utilized and the three squares meet at a single point. A closed formula for determining this and similar solutions would certainly be of interest. In fact, G. A. Katona in [1] and J. B. Kruskal in [3] independently proved a closed formula for the question of creating triangles out of edges, and for other questions concerning faces of certain dimensions of simplicial complexes. As a result of knowing so closely how faces of different dimensions are related in simplicial complexes, a complete characterization of simplicial complexes in terms of the number of possible faces of certain dimension resulted. This characterization is most commonly viewed as concerning the f-vectors of simplicial complexes - the tuples that store the number of faces of each dimension that the corresponding simplicial complexes have. Bernt Lindström proposes a solution in [4] for the question of creating squares out of edges, and all similar questions for cubical complexes which lie in cubes (from this point all cubical complexes discussed are assumed to lie in cubes unless otherwise stated). Lindström's result is correct, but his proof contains a flaw (pp. 248-9). The formulation of the main result in Section 5 depends on defining a total order on the faces of cubical complexes and showing that for the optimality conditions of the main result, it may be assumed that the faces of the cube are lowest possible in the total order. This assumption is labelled the *crucial inclusion* and is presented in Section 3; however, this section contains many technical details (including a correction of the flaw) which the reader may desire to skip in favor of the more concrete results of Sections 4-6. Indeed, as long as the background material of Section 2 is understood, there is no serious loss of coherency from proceeding in this fashion. In the same vein, proofs of several technical lemmas are relegated to the Appendix. The corresponding result for simplicial complexes is reviewed in Section 4, and after the main result is presented in Section 5, a complete characterization of f-vectors of cubical complexes which lie in cubes is derived.

2 Background

Definition. *A cubical complex C is a set of faces of an n-dimensional cube with the property that if $F \in C$ is a face of dimension r, where $0 \le r \le n$, then all lower dimensional faces contained in F are also in C.*

Note that this definition of a cubical complex restricts attention to those complexes which lie in cubes. This imbedding provides a structure by which the optimal number of faces in a cubical complex can be determined. To take advantage of this structure, the labelings of vertices and the definitions of faces of a cubical complex must be made explicit. The 2^n vertices of the n-cube in which a complex is embedded are labelled in a customary fashion; the vertex $(x_1 \ldots x_n)$, where $x_k \in \{0, 1\}$ for all k is labelled v_i, where

$$i = \sum_{k=1}^{n} x_k 2^{n-k}.$$

This labelling allows the following partial order on the vertices. If $v_i = (x_1 \ldots x_n)$ and $v_j = (y_1 \ldots y_n)$, then $v_i \prec v_j$ if $x_k \le y_k$ for all k, and $v_i \ne v_j$. To prove the main result of this paper it is necessary to determine what the faces of a cubical complex are and to define a total ordering on the faces of the n-cube.

Example 1. The object in Figure 1 can be viewed as a cubical complex which sits in the 3-cube (indicated by the dashed lines). The members of the complex are the square $ABDC$; the lines AB, AC, BD, CD, and CE; the vertices A, B, C, D, E, and F; and the empty set. The square is the 2-dimensional face, the lines are the 1-dimensional faces, and the vertices are the 0-dimensional faces of the complex. Note that AB is an explicit member of the complex despite $ABDC$ already having

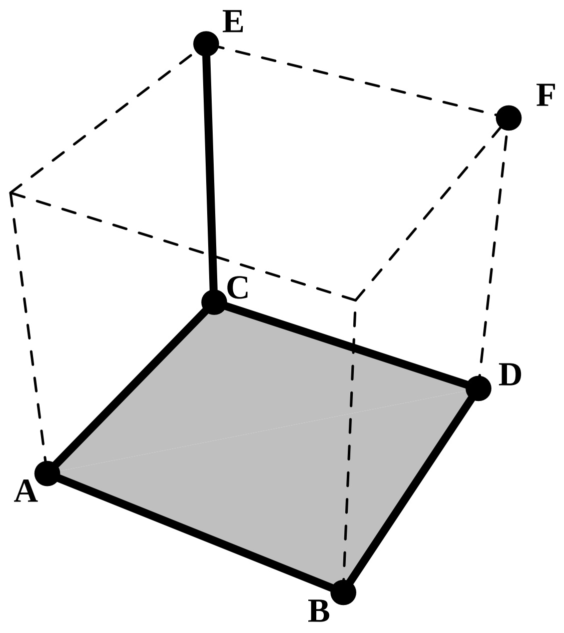

Figure 1: A cubical complex consisiting of 0,1, and 2-dimensional faces sits in the 3-cube.

been given as a member. This satisfies the definition of cubical complexes in that all faces belonging to other faces of higher dimension which are in the complex must also be in the complex. In set notation, the complex in Figure 1 is written as

$$\{ABDC, AB, AC, BD, CD, CE, A, B, C, D, E, F, \emptyset\}.$$

The complex can also be viewed as having been generated by its highest dimensional faces which are not contained in any other faces, written as

$$\langle ABEC, CE, F\rangle.$$

Definition. *A nonempty face of a cubical complex C is denoted w_{ij}, where $v_i = (x_1 \ldots x_n)$ and $v_j = (y_1 \ldots y_n)$ are the vertices of the faces such that $v_i \prec v_k \prec v_j$ for all other vertices in the face. The empty face is denoted by the empty set. The face w_{ij} can be written as $w_{ij} = (x_1 \ldots x_n, y_1 \ldots y_n)$ or as*

$$w_{ij} = \begin{pmatrix} x_1 \ldots x_n \\ y_1 \ldots y_n \end{pmatrix}.$$

The empty set is allowed to be a face, of course, because of closure considerations. Also, it is desireable to be able to represent any vertex $(x_1 \ldots x_n)$ as a 0-dimensional face by writing $w_{ii} = (x_1 \ldots x_n, x_1 \ldots x_n)$. Note that the n-dimensional cube is also the face $w_{02^n} = (0 \ldots 0, 1 \ldots 1)$.

The total ordering on the faces of the n-cube relies upon the labelling of the vertices which determine the faces. If $w_{i_1j_1} \neq w_{i_2j_2}$ are two faces of the n cube, then they are ordered by

$$w_{i_1j_1} < w_{i_2j_2} \qquad \text{if} \quad j_1 < j_2 \qquad \text{or} \qquad j_1 = j_2 \text{ and } i_2 < i_1.$$

Definition. *Let T be a finite set with a total order defined on it. A set $I \subseteq T$ is called initial if I consists of the first $|I|$ elements of T in the total order.*

The idea of an initial wset S_r of r-faces of the n-cube, or the first $|S_r|$ r-faces of the n-cube, will be useful in proving Theorems 1 and 2.

A face w_{ij} has as its subfaces the empty set and all faces w_{gh} such that $i \leq g \leq h \leq j$, since all vertices $v_k \notin \{v_i, v_j\}$ of the face w_{ij} have v_i as an ancestor and v_j as a descendent in the partial ordering on vertices. This is convenient for the total ordering on faces since all nonempty subfaces w_{gh} of w_{ij} satisfy $w_{gh} \leq w_{ij}$. The idea of dimension of a face is now made precise.

Definition. *The dimension of a face $w_{ij} = (x_1 \ldots x_n, y_1 \ldots y_n)$ in a cubical complex is $r = \sum_{k=1}^{n}(y_k - x_k)$; w_{ij} is called an r-face of the cubical complex.*

Example 2. The cubical complex in Figure 1 can be viewed as a set of faces w_{ij} sitting in a 3-dimensional binary lattice. After the vertices are labelled, it is straightforward to label the faces; this new vertex labelling appears in Figure 2. From the previous definitions, the complex can be written in ascending total order as

$$\left\{\emptyset, \begin{pmatrix} 000 \\ 000 \end{pmatrix}, \begin{pmatrix} 001 \\ 001 \end{pmatrix}, \begin{pmatrix} 000 \\ 001 \end{pmatrix}, \begin{pmatrix} 010 \\ 010 \end{pmatrix}, \begin{pmatrix} 000 \\ 010 \end{pmatrix}, \begin{pmatrix} 011 \\ 011 \end{pmatrix},\right.$$

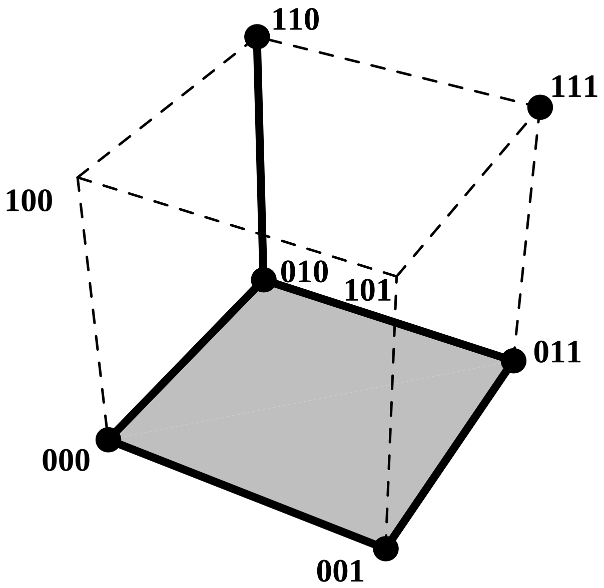

Figure 2: The cubical complex of Figure 1 appears here in the context of a 3-dimensional binary lattice.

$$\begin{pmatrix} 010 \\ 011 \end{pmatrix}, \begin{pmatrix} 001 \\ 011 \end{pmatrix}, \begin{pmatrix} 000 \\ 011 \end{pmatrix}, \begin{pmatrix} 110 \\ 110 \end{pmatrix}, \begin{pmatrix} 010 \\ 110 \end{pmatrix}, \begin{pmatrix} 111 \\ 111 \end{pmatrix} \Bigg\} .$$

If the binary vertices are converted to positive integers in the standard fashion with the most significant bit on the left, the complex in Figure 2 can be written as

$$\{\emptyset, w_{00}, w_{11}, w_{01}, w_{22}, w_{02}, w_{33}, w_{23}, w_{13}, w_{03}, w_{66}, w_{26}, w_{77}\} .$$

Finally, in the notation of a complex generated by its largest faces not contained in other faces, the complex can be written as generated by a 2-face, a 1-face, and a 0-face, respectively:

$$\langle w_{03}, w_{26}, w_{77} \rangle.$$

By definition if a cubical complex C contains an r-face w_{ij}, it also contains all of the $(r-1)$-faces contained in w_{ij}. These faces are called the boundary of w_{ij} and are denoted by ∂w_{ij}, where ∂ is the boundary operator. Furthermore, if S is a set of faces of a cubical complex, then ∂S, the boundary of S, is defined to be the union of the boundaries of each face in S. It is convenient to use S_r to denote the set of r-faces of a cubical complex; whence ∂S_r, the boundary of S_r, is the set of $(r-1)$ faces, each of which is in the boundary of some r-face in S_r. Conversely, $\partial^{-1}S$, the coboundary of S_r, is defined to be the largest set of $(r+1)$-faces, the boundary of which is contained in S_r. In other words, $\partial^{-1}S_r$ is the largest set of $(r+1)$-faces T_{r+1} such that $\partial T_{r+1} \subseteq S_r$. By this definition, the boundary of a vertex (a 0-face) is the empty set. The operators ∂ and ∂^{-1} applied p times each are written respectively as ∂^p (the pth boundary) and ∂^{-p} (the pth coboundary).

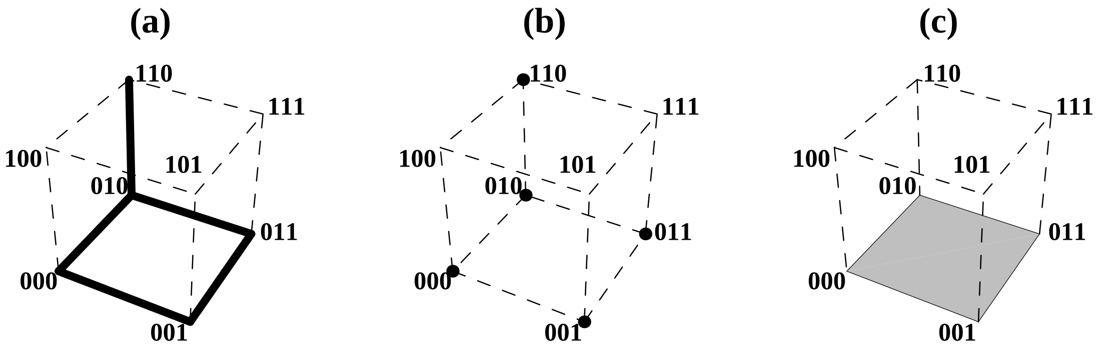

Figure 3: The set of 1-faces from Figure 1, S_1, appears in (a). The boundary set ∂S_1 appears in (b), and the coboundary set $\partial^{-1}S_1$ is represented in (c).

Example 3. Consider the set S_1 of the 1-dimensional faces of the complex in Figure 2. By inspection it is easy to verify the following:

$$S_1 = \{w_{01}, w_{02}, w_{23}, w_{13}, w_{26}\}.$$

The set of faces of S_1 is represented in Figure 3(a). The boundary of S_1, or all the vertices contained in the edges of S_1, is $\{w_{00}, w_{11}, w_{22}, w_{33}, w_{66}\}$, and is represented in Figure 3(b). Verification of $\partial^{-1}\partial S_1 = S_1$ can be done empirically. The coboundary of S_1 is the set of 2-faces, $\{w_{03}\}$, and is represented in Figure 3(c). Note that the boundary of the face in Figure 3(c) is not S_1; no 2-face arises from the 1-face w_{26} in

S_1, since other required 1-faces in the boundary of the 2-face are absent from 3(a). This illustrates empirically the containment $\partial\partial^{-1}S_1 \subseteq S_1$.

The total order on the faces of the n-cube lends an inherited total order to the r-faces of the n-cube. For this reason, if S_r is a set of r-faces, the $|S_r|$ first r-faces in the total order may be considered; this set is called RS_r, where R is the replacement operator. For any face $w_{ij} = (v_i, v_j) = (x_1 \dots x_n, y_1 \dots y_n)$, the three possibilities for (x_k, y_k), for all k, are $(0,0)$, $(0,1)$, and $(1,1)$, since $v_i \prec v_j$ or $v_i = v_j$. The total order on the set of r-faces S_r of the n-cube can be further restricted to r-faces which have the kth coordinate (x_k, y_k), resulting in three disjoint sets which are labelled $S_{r,k}(x_k, y_k)$ where k is fixed and $(x_k, y_k) \in \{(0,0),(0,1),(1,1)\}$. Furthermore, the restricted replacement operator R_k acting on S_r sends $S_{r,k}(x_k, y_k)$ to the $|S_{r,k}(x_k, y_k)|$ lowest r-faces whose kth coordinate is fixed at (x_k, y_k), and returns the union over the three possibilities for (x_k, y_k).

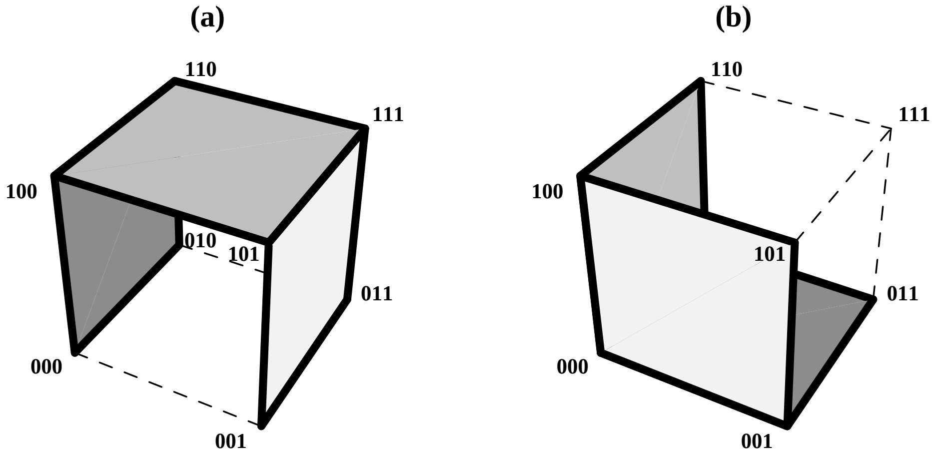

Figure 4: A set of 2-faces, S_2, appears in (a), along with its boundary set of 1-faces. The replacement set, RS_2, appears in (b), along with its boundary set of 1-faces.

Example 4. In a sense, the replacement operator R acts on a set of r-faces by compressing them into the smallest possible configuration (the "smallness" is determined by considering the total order). Figure 4(a) depicts three 2-dimensional faces and the ten 1-dimensional faces of their boundary sitting in the 3-cube. Letting S_2 be the 2-dimensional faces of Figure 4(a) gives $S_2 = \{w_{06}, w_{47}, w_{17}\}$. The boundary of S_2 is $\partial S_2 = \{w_{02}, w_{13}, w_{04}, w_{45}, w_{15}, w_{46}, w_{26}, w_{67}, w_{57}, w_{37}\}$, arranged in ascending total order. Figure 4(b) represents the replacement of S_2, or RS_2. The lowest three 2-faces in the total order are $RS_2 = \{w_{03}, w_{05}, w_{06}\}$. The boundary of RS_2 is $\partial RS_2 = \{w_{01}, w_{02}, w_{23}, w_{13}, w_{04}, w_{45}, w_{15}, w_{46}, w_{26}\}$, arranged in ascending total order. Note that ∂RS_2 is smaller than ∂S_2.

3 The Crucial Inclusion

The result of Lindstöm's paper hinges upon the following theorem involving boundaries of sets of r-faces, which appears as Theorem 1 in [4].

Theorem 1. *Let S_r be a set of r-faces in the n-cube, with $0 \leq r \leq n$. Let ∂ and R be the boundary and replacement operators defined previously. Then*

$$\partial RS_r \subseteq R\partial S_r.$$

This theorem will also be referred to as the crucial inclusion. From this theorem arises a corollary which appears as Corollary 1 in [4], and allows the computations for the main result of this paper in Section 5.

Corollary 1. *$|\partial^p RS_r| \leq |\partial^p S_r|$ and $|\partial^{-p} S_r| \leq |\partial^{-p} RS_r|$, where p is a positive integer.*

Corollary 1 states first that the size of the pth boundary of an initial set of r-faces RS_r is no larger than the size of the pth boundary of the original set of r-faces S_r. If s is the dimension of the faces in $|\partial^p RS_r|$, then the smallest number of s-faces required to compose the boundary set $\partial^p S_r$ is the number of s-faces in the boundary set $\partial^p RS_r$. Thus for the main result, a formula must be established which counts the number of s-faces in the boundary of an initial set RS_r of r-faces. Similarly, the second part of the Corollary determines that the number of s-faces in the pth coboundary set $\partial^{-p}S_r$ is bounded above by the number of s-faces in the pth coboundary set $\partial^{-p}RS_r$. It happens that the above mentioned formula also counts the number of s-faces in the coboundary of an initial set RS_r r-faces.

The proof of this corollary, which appears in [4], follows the proof of Theorem 1.

Proof of Theorem 1. The proof of Theorem 1 has two parts. First, it is necessary to show that the size of ∂RS_r is no greater than the size of ∂S_r. From there, it is sufficient to show that ∂RS_r is an initial set of $(r-1)$-faces, as by definition $R\partial S_r$ is an initial set of $(r-1)$-faces.

The proof of $|\partial RS_r| \leq |\partial S_r|$ is by induction on the dimension n and occurs in two stages ($n = 1$ and $n = 2$ have a small number of cases for which the theorem can be verified in a straightforward manner). First, the restricted operator R_k is applied to S_r for varying values of k until the resulting set of r-faces is constant. It must be shown that the resulting set has a boundary that is no larger than that of S_r. This is done by showing that during each restricted replacement step the size of the boundary of the set of r-faces does not increase. Second, since the resulting set in general is not equal to RS_r, it must be shown that the size of the boundary of RS_r is no larger than the size of the boundary of the resulting set.

3.1 The Sequential Replacement

Since there is a total ordering on the faces of the n-cube, each face can be associated with a positive integer; i.e., there is a mapping θ from the faces of the n-cube to a finite subset of the positive integers which preserves the total order on the faces, giving $\theta(w_{gh}) < \theta(w_{ij})$ whenever w_{gh} and w_{ij} are faces such that $w_{gh} < w_{ij}$ in the total order. Furthermore, for a set of r-faces S_r, define $sum(S_r)$ to be the sum of the numbers associated with the faces of S_r, or

$$sum(S_r) = \sum_{w_{ij} \in S_r} \theta(w_{ij}).$$

As the restricted replacement operator R_k may only replace r-faces of S_r with r faces that are lower in the total order, it follows that $sum(R_k S_r) \leq sum(S_r)$ for any index k fixed between 1 and n. This fact determines that the sequence constructed below will eventually become constant, as it is monotonically decreasing and bounded below by $sum(RS_r)$.

If S_r is an arbitrary set of r-faces of the n-cube, apply the restricted replacement operators $R_1, R_2, \ldots, R_n, R_1, \ldots$ repeatedly to obtain the infinite sequence

$$S_r^0 = S_r, S_r^1 = R_1 S_r^0, \ldots, S_r^n = R_n S_r^{n-1}, S_r^{n+1} = R_1 S_r^n, \ldots.$$

If $S_r^{l+1} \neq S_r^l$ it follows that $sum(S_r^{l+1}) < sum(S_r^l)$. It is not difficult to see that if at some point in the sequence the set S_r^l is constant under the application of $R_1, R_2, \ldots, R_n$, then the sequence itself will be constant from that set onward. This establishes the existence of an index q such that $S_r^q = S_r^{q+1} = \ldots = S_r^{q+n} = \ldots$.

Example 5. Let S_1 be the 1-faces of the cubical complex in Figure 5(a). Then

$$S_1 = \left\{ \begin{pmatrix} 001 \\ 011 \end{pmatrix}, \begin{pmatrix} 010 \\ 110 \end{pmatrix}, \begin{pmatrix} 110 \\ 111 \end{pmatrix}, \begin{pmatrix} 101 \\ 111 \end{pmatrix} \right\}$$

(all sets are written in increasing total order). Now consider $S_1 = S_1^0$ as the first set in the sequence of fixed coordinate replacements. The next set in the sequence, $R_1 S_1^0 = S_1^1$, is depicted in Figure 5(b). The operator R_1 acts on S_1 by fixing the first coordinate. The face $\begin{pmatrix} 001 \\ 011 \end{pmatrix}$ of S_1 is replaced with the face $\begin{pmatrix} 000 \\ 001 \end{pmatrix}$, which is the lowest 1-face in the total order with the first coordinate fixed at $(0,0)$. Similiary, $\begin{pmatrix} 110 \\ 111 \end{pmatrix}$ and $\begin{pmatrix} 101 \\ 111 \end{pmatrix}$ are replaced by $\begin{pmatrix} 100 \\ 100 \end{pmatrix}$ and $\begin{pmatrix} 101 \\ 110 \end{pmatrix}$. Finally, $\begin{pmatrix} 010 \\ 110 \end{pmatrix}$ is replaced by $\begin{pmatrix} 000 \\ 100 \end{pmatrix}$. The set

$$S_1^1 = \left\{ \begin{pmatrix} 000 \\ 001 \end{pmatrix}, \begin{pmatrix} 000 \\ 100 \end{pmatrix}, \begin{pmatrix} 100 \\ 101 \end{pmatrix}, \begin{pmatrix} 100 \\ 110 \end{pmatrix} \right\}$$

is obtained, as shown in Figure 5(b). The next step in the sequential replacement is to fix the 2nd coordinate, finding the set $R_2 S_1^1 = S_1^2$. This set is represented in Figure 5(c); i. e. ,

$$S_1^2 = \left\{ \begin{pmatrix} 000 \\ 001 \end{pmatrix}, \begin{pmatrix} 000 \\ 010 \end{pmatrix}, \begin{pmatrix} 000 \\ 100 \end{pmatrix}, \begin{pmatrix} 100 \\ 101 \end{pmatrix} \right\}.$$

The sequence becomes constant with the fourth member, $R_3 S_1^2 = S_1^3$, depicted in Figure 5(d); i. e. ,

$$S_1^3 = \left\{ \begin{pmatrix} 000 \\ 001 \end{pmatrix}, \begin{pmatrix} 000 \\ 010 \end{pmatrix}, \begin{pmatrix} 010 \\ 011 \end{pmatrix}, \begin{pmatrix} 000 \\ 100 \end{pmatrix} \right\}.$$

Note that S_1^3 is not an initial set of 1-faces of the 3-cube, because $w_{13} < w_{04}$. Also, the number of faces (vertices) of the boundary does not increase as the sequence progresses.

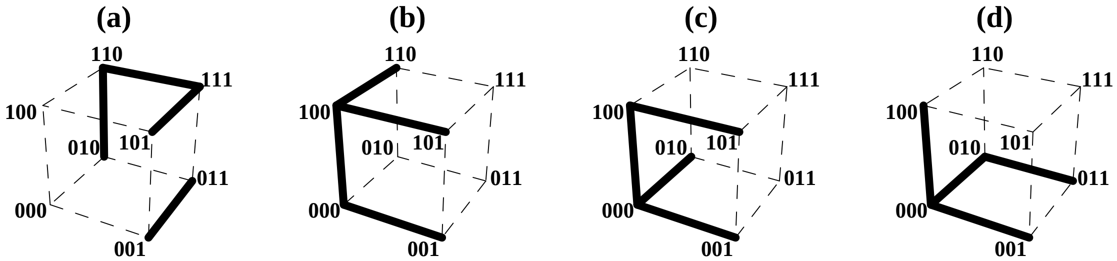

Figure 5: A set, S_1, of 1-faces of the 3-cube appears in (a); (a)-(d) are the first four members of the restricted replacement sequence $S_1 = S_1^0, R_1 S_1^0 = S_1^1, R_2 S_1^1 = S_1^2, R_3 S_1^2 = S_1^3, \ldots$, which becomes constant after the third member.

To show $|\partial S_r^q| \leq |\partial S_r|$ it is sufficient to show that for each index l, $|\partial S_r^{l+1}| \leq |\partial S_r^l|$; in other words, the size of the boundary set does not increase along the sequence. Without loss of generality, l can be assumed to be less than n, since the issue of interest is the coordinate fixed by the restricted replacement operator; the set S_r^{l+1} is obtained from the set S_r^l by applying the restricted replacement operator $R_{(l \bmod n)+1}$ to S_r^l.

Given two adjacent sets in the sequence which may be denoted as S_r^{l-1} and $R_l S_r^{l-1}$, showing that the number $(r-1)$-faces of the boundary does not increase is equivalent to showing $|\partial R_l S_r^{l-1}| \leq |\partial S_r^{l-1}|$, which is equivalent to showing

$$\begin{aligned} & |\partial R_l S_{r,l}^{l-1}(0,0) \cup \partial R_l S_{r,l}^{l-1}(0,1) \cup \partial R_l S_{r,l}^{l-1}(1,1)| \\ \leq \quad & |\partial S_{r,l}^{l-1}(0,0) \cup \partial S_{r,l}^{l-1}(0,1) \cup \partial S_{r,l}^{l-1}(1,1)|. \end{aligned}$$

By inclusion-exclusion, this is equivalent to showing the following:

$$\begin{aligned} & |\partial R_l S_{r,l}^{l-1}(0,0)| + |\partial R_l S_{r,l}^{l-1}(0,1)| + |\partial R_l S_{r,l}^{l-1}(1,1)| - \\ & |\partial R_l S_{r,l}^{l-1}(0,0) \cap \partial R_l S_{r,l}^{l-1}(0,1)| - |\partial R_l S_{r,l}^{l-1}(0,1) \cap \partial R_l S_{r,l}^{l-1}(1,1)| - \\ & |\partial R_l S_{r,l}^{l-1}(0,0) \cap \partial R_l S_{r,l}^{l-1}(1,1)| + \\ & |\partial R_l S_{r,l}^{l-1}(0,0) \cap \partial R_l S_{r,l}^{l-1}(0,1) \cap \partial R_l S_{r,l}^{l-1}(1,1)| \\ \leq \quad & |\partial S_{r,l}^{l-1}(0,0)| + |\partial S_{r,l}^{l-1}(0,1)| + |\partial S_{r,l}^{l-1}(1,1)| - \\ & |\partial S_{r,l}^{l-1}(0,0) \cap \partial S_{r,l}^{l-1}(0,1)| - |\partial S_{r,l}^{l-1}(0,1) \cap \partial S_{r,l}^{l-1}(1,1)| - \\ & |\partial S_{r,l}^{l-1}(0,0) \cap \partial S_{r,l}^{l-1}(1,1)| + \\ & |\partial S_{r,l}^{l-1}(0,0) \cap \partial S_{r,l}^{l-1}(0,1) \cap \partial S_{r,l}^{l-1}(1,1)|. \end{aligned} \tag{1}$$

Lindström shows in [4] that the following equations hold:

$$\begin{aligned} |\partial R_l S_{r,l}^{l-1}(0,0)| &\leq |\partial S_{r,l}^{l-1}(0,0)| \\ |\partial R_l S_{r,l}^{l-1}(0,1)| &\leq |\partial S_{r,l}^{l-1}(0,1)| \\ |\partial R_l S_{r,l}^{l-1}(1,1)| &\leq |\partial S_{r,l}^{l-1}(1,1)| \end{aligned} \tag{2}$$

by an argument which invokes the induction hypothesis on the original theorem (recall that induction is on the dimension, n, of the n-cube in which all of the faces are imbedded). Of course, (2) does not necessarily imply (1). The proof of this implication, omitted in [4], is included here.

First, it can be shown that two terms in either side of (1) vanish; i. e. ,

$$|\partial R_l S_{r,l}^{l-1}(0,0) \cap \partial R_l S_{r,l}^{l-1}(0,1) \cap \partial R_l S_{r,l}^{l-1}(1,1)| \quad = \quad 0$$

$$\begin{aligned}
|\partial R_l S_{r,l}^{l-1}(0,0) \cap \partial R_l S_{r,l}^{l-1}(1,1)| &= 0 \\
|\partial S_{r,l}^{l-1}(0,0) \cap \partial S_{r,l}^{l-1}(0,1) \cap \partial S_{r,l}^{l-1}(1,1)| &= 0 \\
|\partial S_{r,l}^{l-1}(0,0) \cap \partial S_{r,l}^{l-1}(1,1)| &= 0.
\end{aligned}$$

This is because all $(r-1)$-faces of the boundary set $\partial R_l S_{r,l}^{l-1}(0,0)$ must have $(0,0)$ in the lth coordinate, and all $(r-1)$-faces of the boundary set $\partial R_l S_{r,l}^{l-1}(1,1)$ must have $(1,1)$ as the lth coordinate, making the two boundary sets disjoint. Similarly, $\partial S_{r,l}^{l-1}(0,0) \cap \partial S_{r,l}^{l-1}(1,1) = \emptyset$. From this the reduced equation from (1) is obtained:

$$\begin{aligned}
& |\partial R_l S_{r,l}^{l-1}(0,0)| + |\partial R_l S_{r,l}^{l-1}(0,1)| + |\partial R_l S_{r,l}^{l-1}(1,1)| - \\
& |\partial R_l S_{r,l}^{l-1}(0,0) \cap \partial R_l S_{r,l}^{l-1}(0,1)| - |\partial R_l S_{r,l}^{l-1}(0,1) \cap \partial R_l S_{r,l}^{l-1}(1,1)| \\
\leq\ & |\partial S_{r,l}^{l-1}(0,0)| + |\partial S_{r,l}^{l-1}(0,1)| + |\partial S_{r,l}^{l-1}(1,1)| - \\
& |\partial S_{r,l}^{l-1}(0,0) \cap \partial S_{r,l}^{l-1}(0,1)| - |\partial S_{r,l}^{l-1}(0,1) \cap \partial S_{r,l}^{l-1}(1,1)|. \qquad (3)
\end{aligned}$$

Furthermore, all the $(r-1)$-faces of the boundary set $\partial S_{r,l}^{l-1}(0,0)$ have $(0,0)$ as the lth coordinate; all the $(r-1)$-faces of the intersection $\partial S_{r,l}^{l-1}(0,0) \cap \partial S_{r,l}^{l-1}(0,1)$ must be obtained from r-faces of $S_{r,l}^{l-1}(0,1)$ which have the lth coordinate changed from $(0,1)$ to $(0,0)$. This is exactly the set $(\partial_l(0,0))\, S_{r,l}^{l-1}(0,1)$, where $\partial_l(x_l, v_l) S_r$ is defined to be all $(r-1)$-faces of the boundary set ∂S_r which have the lth coordinate fixed at (x_l, y_l). By this and similar reasoning, the following containments are obtained:

$$\begin{aligned}
\partial S_{r,l}^{l-1}(0,0) \cap \partial S_{r,l}^{l-1}(0,1) &\subseteq (\partial_l(0,0))\, S_{r,l}^{l-1}(0,1) \\
\partial S_{r,l}^{l-1}(1,1) \cap \partial S_{r,l}^{l-1}(0,1) &\subseteq (\partial_l(1,1))\, S_{r,l}^{l-1}(0,1) \\
\partial R_l S_{r,l}^{l-1}(0,0) \cap \partial R_l S_{r,l}^{l-1}(0,1) &\subseteq (\partial_l(0,0))\, R_l S_{r,l}^{l-1}(0,1) \\
\partial R_l S_{r,l}^{l-1}(1,1) \cap \partial R_l S_{r,l}^{l-1}(0,1) &\subseteq (\partial_l(1,1))\, R_l S_{r,l}^{l-1}(0,1). \qquad (4)
\end{aligned}$$

Finally, for each of the following two pairs of sets, it can be shown that one set is a subset of the other.

$$\begin{aligned}
&\{\partial R_l S_{r,l}^{l-1}(0,0), \quad \partial_l(0,0) R_l S_{r,l}^{l-1}(0,1)\} \\
&\{\partial R_l S_{r,l}^{l-1}(1,1), \quad \partial_l(1,1) R_l S_{r,l}^{l-1}(0,1)\} \qquad (5)
\end{aligned}$$

By considering the pair of sets $\partial R_l S_{r,l}^{l-1}(0,0)$ and $(\partial_l(0,0))\, R_l S_{r,l}^{l-1}(0,1)$ it can be seen that the set $R_l S_{r,l}^{l-1}(0,0)$ is the $|S_{r,l}^{l-1}(0,0)|$ first r-faces of the n-cube which have the lth coordinate fixed at $(0,0)$. If the lth position is ignored, the $|S_{r,l}^{l-1}(0,0)|$ first r-faces of the $(n-1)$-cube are obtained. By induction on the main theorem, the boundary of this set of r-faces of the $(n-1)$-cube is an initial set of $(r-1)$-faces of the $(n-1)$-cube. Restoring the lth coordinate of these $(r-1)$-faces at $(0,0)$ gives $\partial R_l S_{r,l}^{l-1}(0,0)$, which exactly correspond to the first $|\partial R_l S_{r,l}^{l-1}(0,0)|$ $(r-1)$-faces of the n-cube which have the lth coordinate fixed at $(0,0)$. The set $(\partial_l(0,0))\, R_l S_{r,l}^{l-1}(0,1)$ is all of the $(r-1)$-faces of $\partial R_l S_{r,l}^{l-1}(0,1)$ which have had the lth coordinate changed from $(0,1)$ to $(0,0)$. This corresponds to the $|R_l S_{r,l}^{l-1}(0,1)|$ first $(r-1)$-faces in the n-cube which have the lth coordinate fixed at $(0,0)$. Both of these sets are initial sets of $(r-1)$-facesof the n-cube with the lth coordinate fixed at $(0,0)$; hence one set is a subset of the other. Containment for the second pair of sets is proved similarly.

With the previous information the inequality in (1) can now be proved via the simplification in (3).

(4) implies

$$\begin{aligned}|\partial S_{r,l}^{l-1}(0,0) \cap \partial S_{r,l}^{l-1}(0,1)| &\leq |S_{r,l}^{l-1}(0,1)|\\ |\partial S_{r,l}^{l-1}(1,1) \cap \partial S_{r,l}^{l-1}(0,1)| &\leq |S_{r,l}^{l-1}(0,1)|;\end{aligned}$$

(5) implies

$$\begin{aligned}|\partial R_l S_{r,l}^{l-1}(0,0) \cap \partial R_l S_{r,l}^{l-1}(0,1)| &= \min\{|\partial R_l S_{r,l}^{l-1}(0,0)|, |S_{r,l}^{l-1}(0,1)|\}\\ |\partial R_l S_{r,l}^{l-1}(1,1) \cap \partial R_l S_{r,l}^{l-1}(0,1)| &= \min\{|\partial R_l S_{r,l}^{l-1}(1,1)|, |S_{r,l}^{l-1}(0,1)|\}.\end{aligned}$$

From these, four cases for the inequality in (3) are obtained. In the first case

$$\begin{aligned}|\partial R_l S_{r,l}^{l-1}(0,0) \cap \partial R_l S_{r,l}^{l-1}(0,1)| &= |\partial R_l S_{r,l}^{l-1}(0,0)| \text{ and}\\ |\partial R_l S_{r,l}^{l-1}(1,1) \cap \partial R_l S_{r,l}^{l-1}(0,1)| &= |\partial R_l S_{r,l}^{l-1}(1,1)|\end{aligned}$$

imply (3) by the following:

$$\begin{aligned}
& |\partial R_l S_{r,l}^{l-1}(0,0)| + |\partial R_l S_{r,l}^{l-1}(0,1)| + |\partial R_l S_{r,l}^{l-1}(1,1)| - \\
& |\partial R_l S_{r,l}^{l-1}(0,0) \cap \partial R_l S_{r,l}^{l-1}(0,1)| - |\partial R_l S_{r,l}^{l-1}(0,1) \cap \partial R_l S_{r,l}^{l-1}(1,1)| \\
= \; & |\partial R_l S_{r,l}^{l-1}(0,1)| \\
\leq \; & |\partial S_{r,l}^{l-1}(0,1)| \qquad \text{by (2)} \\
\leq \; & |\partial S_{r,l}^{l-1}(0,0)| + |\partial S_{r,l}^{l-1}(0,1)| + |\partial S_{r,l}^{l-1}(1,1)| - \\
& |\partial S_{r,l}^{l-1}(0,0) \cap \partial S_{r,l}^{l-1}(0,1)| - |\partial S_{r,l}^{l-1}(0,1) \cap \partial S_{r,l}^{l-1}(1,1)|,
\end{aligned}$$

since the following equations hold:

$$\begin{aligned}
|\partial S_{r,l}^{l-1}(0,0) \cap \partial S_{r,l}^{l-1}(0,1)| & \leq |\partial S_{r,l}^{l-1}(0,0)| \\
|\partial S_{r,l}^{l-1}(0,1) \cap \partial S_{r,l}^{l-1}(1,1)| & \leq |\partial S_{r,l}^{l-1}(1,1)|.
\end{aligned}$$

The other cases give the desired result similarly and are omitted.

In conclusion, by (1), it is ascertained that $|\partial R_l S_r^{l-1}| \leq |\partial S_r^{l-1}|$ for arbitrary l between 1 and n. Repeated application of this fact shows that the size of the boundary set is monotonically decreasing along the sequence; it follows that $|\partial S_r^q| \leq |\partial S_r|$, as desired.

3.2 The Arbitrary Replacement

After the sequential replacement has been done, for which the size of the boundary of the set of r-faces does not increase, an arbitrary replacement is done to obtain the initial set of size $|S_r|$ of r-faces, since in general, $RS_r \neq S_r^q$. The arbitrary replacement is done by replacing the largest r-face (a, a') in S_r^q with the smallest r-face (b, b') not in S_r^q iteratively until $RS_r^q = RS_r$ is obtained. Once again, it must be ascertained that the size of the boundary set of these r-faces does not increase under this arbitrary replacement. This is achieved by showing that the only possible candidates for the two faces (a,a') and (b,b') require $|\partial((S_r^q - \{(a,a')\}) \cup \{(b,b')\})| \leq |\partial S_r^q|$. A short argument on the iteration of this step results in $|\partial RS_r| \leq |\partial S_r^q|$.

Example 6. Consider the sequential replacement on the set S_1 in Example 5. The constant result of the sequential replacement, S_1^3, is not the four lowest 1-faces of the 3-cube. It has been shown that the size of the boundary set does not increase along the restricted replacement sequence, but to satisfy the crucial inclusion, it must also hold that S_1^3 does not have a smaller boundary set than RS_1 (which is also RS_1^3). There are two differences between the sets S_1^3 and RS_1, which are shown as Figure 6(a) and (b), respectively. S_1^3 contains the face $\begin{pmatrix} a \\ a' \end{pmatrix} = \begin{pmatrix} 000 \\ 100 \end{pmatrix}$, and RS_1 does not. RS_1 contains the face $\begin{pmatrix} b \\ b' \end{pmatrix} = \begin{pmatrix} 001 \\ 011 \end{pmatrix}$, and S_1^3 does not.

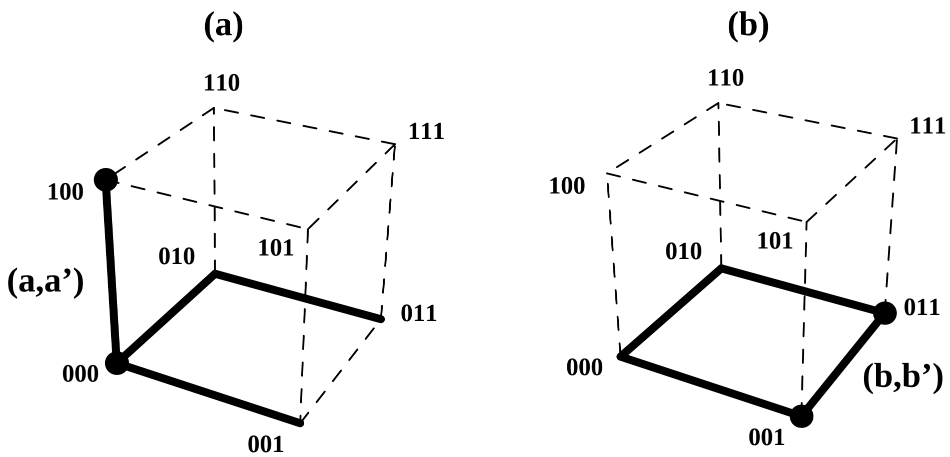

Figure 6: The set S_1^3, from Example 5 and the set RS_1 are depicted in (a) and (b), respectively, with selected boundary elements.

Now consider the effect of removing (a,a') from S_1^3 and adding (b,b') in its place. The boundaries of (a,a') and (b,b') are denoted in Figure 6. Removing (a,a') from S_1^3 removes the boundary vertex $\begin{pmatrix} 100 \\ 100 \end{pmatrix}$. The vertices in the boundary of (b,b') are already contained in the boundary set $\partial(S_1^3 - \{(a,a')\})$; hence no boundary elements are added when the face (b,b') is added to $S_1^2 - \{(a,a')\}$. Since $(S_1^3 -$

$\{(a,a')\}) \cup \{(b,b')\} = RS_1$, S_1^3 a larger boundary set than RS_1. The required inequality, $|\partial RS_1| \leq |\partial S_1|$, is satisfied.

Recall that S_r^q is the set of r-faces obtained from S_r by the sequential replacement step. Let (a,a') be the largest r-face in S_r^q and let (b,b') be the smallest r-face no in S_r^q. To show $|\partial RS_r| \leq |\partial S_r^q| \leq |\partial S_r|$, it is sufficient to show that whenever $S_r^q \neq RS_r$, replacing (a,a') with (b,b') does not increase the size of the boundary of the resultant set of r-faces. The two cases that arise are $a' = b'$ and $a' > b'$.

Case 1. Let $a' = b'$. Then $a' = b' = (1\ldots 1)$, or else there is an index j for which $a'_j = b'_j = 0$. Then $a_j = b_j = 0$ since $(1,0)$ pairs are illegal, and $(a_j, a'_j) = (b_j, b'_j) = (0,0)$, and (a,a') would have been replaced with (b,b') in the sequential replacement step. Furthermore, the following properties must hold:

1. $a_i \neq b_i$ for all $i \in \{1\ldots n\}$; otherwise (a,a') (or some other r-face) would have been replaced with (b,b') in the sequential replacement step.

2. $a_1 = 0$ and $b_1 = 1$; otherwise $a < b$ and $(a,a') < (b,b')$, which violates the assumed order on (a,a') and (b,b').

3. n is even and $r = n/2$; otherwise, without loss of generality, there are more $(1,1)$ coordinates than $(0,1)$ coordinates of both (a,a') and (b,b'); however, in this case there must be an index j for which $(a_i, a'_i) = (b_i, b'_i) = (1,1)$. Then (a,a') would have been replaced with (b,b') in the sequential replacement step.

In particular, if $n = 2$, the above conditions require $\begin{pmatrix} b \\ b' \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$ and $\begin{pmatrix} a \\ a' \end{pmatrix} = \begin{pmatrix} 01 \\ 11 \end{pmatrix}$. This gives $\partial\{(b,b')\} = \left\{\begin{pmatrix} 10 \\ 10 \end{pmatrix}, \begin{pmatrix} 11 \\ 11 \end{pmatrix}\right\}$ and $\partial\{(a,a')\} = \left\{\begin{pmatrix} 01 \\ 01 \end{pmatrix}, \begin{pmatrix} 11 \\ 11 \end{pmatrix}\right\}$. But the boundary $\begin{pmatrix} 10 \\ 10 \end{pmatrix}$ also comes from the r-face $\begin{pmatrix} 00 \\ 10 \end{pmatrix}$; hence $\partial\left((S_r^q - \{(a,a')\}) \cup \{(b,b')\}\right) \leq \partial S_r^q$, as desired, since $\begin{pmatrix} 00 \\ 10 \end{pmatrix} \in S_r^q$ by assumption.

For the case $n > 2$, i.e., $n \geq 4$, except for the first coordinate of (a, a') which has been shown to be $(0,1)$, there can be no $(0,1)$ coordinates to the left of $(1,1)$ coordinates. Hence

$$\begin{aligned} \begin{pmatrix} a \\ a' \end{pmatrix} &= \begin{pmatrix} 01\ldots10\ldots0 \\ 11\ldots11\ldots1 \end{pmatrix} \text{ and} \\ \begin{pmatrix} b \\ b' \end{pmatrix} &= \begin{pmatrix} 10\ldots01\ldots1 \\ 11\ldots11\ldots1 \end{pmatrix}. \end{aligned}$$

Otherwise, if there exist indices $2 \leq j < k < n$ for which $(a_j, a'_j) = (0,1)$ and $(a_k, a'_k) = (1,1)$, define (d, d') by $(d_j, d'_j) = (1,1)$, $(d_k, d'_k) = (0,1)$ and $(d_i, d'_i) = (a_i, a'_i)$ for all other indices $i \in \{1, \ldots, n\}$ to demonstrate a face which satisfies $(b, b') < (d, d') < (a, a')$; hence, the sequential replacement step would have replaced (a, a') (or some other face) with (b, b'), which is a contradiction.

Now it will be shown that the size of the boundary of $S_r^q - \{(a, a')\}$ does not increase when (b, b') is added. Elements of the boundary of (b, b') are obtained by changing $(0,1)$ pairs to either $(1,1)$ or $(0,0)$; thus fix an index $l \in \{2, \ldots, \frac{n}{2} + 1\}$ whose coordinate is to be changed.

1. If (b_l, b'_l) is changed to $(1,1)$, define the face (c, c') by $(c_l, c'_l) = (1,1)$, $(c_n, c'_n) = (0,1)$ and $(c_i, c'_i) = (b_I, b'_i)$ for all other indices $i \in \{1, \ldots, n\}$. Then (c, c') contains the same boundary element and is an r-face smaller than (b, b'), since $c > b$; hence (c, c') is in R_s^q by definition of (b, b'), and and this boundary element is not added to $\partial(S_r^q - \{(a, a')\})$ when (b, b') is added to $S_r^q - \{(a, a')\})$.

2. Similarly, if (b_l, b'_l) is changed to $(0,0)$, no boundary element is added to $\partial(S_r^q - \{(a, a')\})$ when (b, b') is added to $S_r^q - \{(a, a')\})$.

In either instance, $(S_r^q - \{(a, a')\}) \cup \{(b, b')\}$ has no larger a boundary than S_r^q since removing (a, a') from the set can only take away boundary elements and it has been shown that adding (b, b') to the set does not increase the number of boundaries.

Case 2. Let $a' > b'$. As Lindström shows in [4], there can be only one index l for which $a'_l > b'_l$. For all other indices i, $a'_i \leq b'_i$. Otherwise, there are two indices $1 \leq j < k \leq n$ for which $a'_j > b'_j$ and $a'_k > b'_k$. Without loss of generality, $n > 2$, since otherwise (a, a') and (b, b') are vertices with empty boundary sets. For these two indices, $(b_j, b'_j) = (b_k, b'_k) = (0, 0)$. If $(a_k, a'_k) = (1, 1)$, define the face (d, d') by $(d_k, d'_k) = (a_k, a'_k)$ and $(d_i, d'_i) = (b_i, b'_i)$ for all other indices $i \in \{1, \ldots, n\}$. Then (d, d') is an r-face which satisfies the inequality $(b, b') < (d, d') < (a, a')$, and since (d, d') has coordinates in common with both (a, a') and (b, b'), this is a contradiction since the sequential replacement would have replaced (a, a') with (b, b'). Similarly, if $(a_k, a'_k) = (0, 1)$, choose an index m for which $(b, b') = (0, 1)$ and define (d, d') as before except $(d_m, d'_m) = (1, 1)$ in order to keep the dimension of the face (d, d') fixed at r and obtain the same contradiction through the sequential replacement.

From this point in this case, Lindström's proof is flawed, and is replaced here with a new proof. Furthermore, as in the previous case (a, a') may not have a $(0, 1)$ pair to the left of a $(1, 1)$ pair; i.e., there cannot be $1 \leq j < k \leq n$ for which $(a_j, a'_j) = (0, 1)$ and $(a_k, a'_k) = (1, 1)$. The cases for $n = 2$ are few and easily verified. Otherwise, for $n > 2$, a contradiction is obtained by defining (d, d') as follows: $(d_j, d'_j) = (1, 1)$, $(d_k, d'_k) = (0, 1)$, and $(d_i, d'_i) = (a_i, a'_i)$ for all other indices i. The following configuration is obtained for (a, a'), (b, b'), and (d, d'):

$$\begin{array}{ccc}
\begin{pmatrix} a \\ a' \end{pmatrix} & = & \begin{pmatrix} \cdots & & 0 & & \cdots & & 1 & & \cdots \\ \cdots & & 1 & & \cdots & & 1 & & \cdots \end{pmatrix} \\
\begin{pmatrix} b \\ b' \end{pmatrix} & = & \begin{pmatrix} \cdots & 1 & \vert & 0 & \cdots & 0 & \vert & 0 & \cdots \\ \cdots & 1 & \vert & 0 & \cdots & 1 & \vert & 0 & \cdots \end{pmatrix} \\
\begin{pmatrix} d \\ d' \end{pmatrix} & = & \begin{pmatrix} \cdots & & 1 & & \cdots & & 0 & & \cdots \\ \cdots & & 1 & & \cdots & & 1 & & \cdots \end{pmatrix} \\
 & & \cdots \quad j^{th} coord. \quad \cdots \quad k^{th} coord. \quad \cdots
\end{array}.$$

It is possible for (b_j, b'_j) to be $(1, 1)$ or $(0, 0)$ when $(a_j, a'_j) = (0, 1)$, and (b_k, b'_k) can be $(0, 1)$ or $(0, 0)$ when $(a_k, a'_k) = (1, 1)$. The r-face (d, d') satisfies $(b, b') < (d, d') < (a, a')$ and has $n - 2$ coordinates which match corresponding coordinates of

(a, a'). Also, since there is at most one index l for which $a'_l > b'_l$, (b_j, b'_j) and (b_k, b'_k) cannot both be $(0,0)$, hence either the j^{th} or k^{th} coordinate of (b, b') matches the corresponding coordinate of (d, d'), which is a contradiction, since the sequential replacement would have replaced (a, a') with (b, b').

The face (a, a') is now known to have all $(1,1)$ pairs to the left of all $(0,1)$ pairs. It can be shown that removing (a, a') from S^q_r causes the boundary set of S^q_r to decrease by at least r $(r-1)$-faces. Consider a boundary (e, e') of (a, a') which is obtained by changing the $(0,1)$ pair at the jth coordinate of (a, a') to $(1,1)$. Any other r-face (c, c') distinct from (a, a'), satisfying $(e, e') \in \partial\{(c, c')\}$, must come from, at the kth coordinate of (e, e'), changing either a $(1,1)$ pair or a $(0,0)$ pair to $(0,1)$ (since (e, e') is an $(r-1)$-face, (c, c') must get another $(0,1)$ pair in order to become an r-face). The case shown below is for $(e_k, e'_k) = (1,1)$. Note that for all indices $i \notin \{k, j\}$, $(a_i, a'_i) = (e_i, e'_i) = (c_i, c'_i)$.

$$\begin{array}{ccc}
\begin{pmatrix} a \\ a' \end{pmatrix} & = & \begin{pmatrix} \ldots 1 \ldots 0 \ldots \\ \ldots 1 \ldots 1 \ldots \end{pmatrix} \\
\begin{pmatrix} e \\ e' \end{pmatrix} & = & \begin{pmatrix} \ldots 1 \ldots 1 \ldots \\ \ldots 1 \ldots 1 \ldots \end{pmatrix} \\
\begin{pmatrix} c \\ c' \end{pmatrix} & = & \begin{pmatrix} \ldots 0 \ldots 1 \ldots \\ \ldots 1 \ldots 1 \ldots \end{pmatrix} \\
 & & \ldots k \ldots j \ldots
\end{array}$$

Note that $k \neq j$. Otherwise, if $(e_k, e'_k) = (1,1)$, $k = j$ contradicts the distinctness of (a, a') and (c, c'); in fact, since all $(1,1)$ pairs are to the left of all $(0,1)$ pairs in (a, a'), $k < j$ is obtained. Also, $(e_k, e'_k) = (0,0)$ with $k = j$ contradicts the construction of (e, e') since $(e_j, e'_j) = (1,1)$. If $(e_k, e'_k) = (0,0)$, then $c' > a'$. In either case, $(c, c') > (a, a')$; the r boundary elements of ∂S^q_r created in this manner are only in the boundary set of (a, a') and in the boundary sets of r-faces larger than (a, a'). Since (a, a') is the largest r-face in S^q_r, at least r boundary elements are removed from ∂S^q_r when (a, a') is removed from S^q_r.

Finally, consider adding the face (b, b') to $S^q_r - \{(a, a')\}$. The case $n = 1$ requires

$r = 0$ since otherwise $(a, a') = (b, b')$; hence there are no boundary elements to consider. It is not hard to check that the only case for $n = 2$ is $\begin{pmatrix} a \\ a' \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$ and $\begin{pmatrix} b \\ b' \end{pmatrix} = \begin{pmatrix} 00 \\ 10 \end{pmatrix}$. Since the boundary element $\begin{pmatrix} e \\ e' \end{pmatrix} = \begin{pmatrix} 00 \\ 00 \end{pmatrix} \in \partial\{(b, b')\}$ is also contained in the boundary of the r-face $\begin{pmatrix} 00 \\ 01 \end{pmatrix}$, which is smaller than (b, b'), adding (b, b') to $S^q_r - \{(a, a')\}$ adds at most $r = 1$ boundary elements to $\partial(S^q_r - \{(a, a')\})$. In this case r must be 1; the subcase $r = 0$ has no boundary elements to check, and the subcase $r = 2$ implies $(a, a') = (b, b')$, a contradiction. For the case where $n > 2$, if $r = 0$ there is nothing to check. If $r > 0$, at some coordinate j, $(b_j, b'_j) = (1, 1)$. Otherwise, if there are no $(1, 1)$ pairs in (b, b'), since every $(0, 1)$ pair of (a, a') must then match $(0, 0)$ pairs of (b, b'), $r = 1$; this is because there can only be one index i for which $b'_i < a'_i$. The $(0, 0)$ pairs and $(1, 1)$ pairs of (a, a') must match $(0, 1)$ pairs of (b, b') since matching a $(1, 1)$ pair of (a, a') with a $(0, 0)$ pair of (b, b') would give another index i for which $b'_i < a'_i$, and by assumption there are no $(1, 1)$ pairs of (b, b') to match with the $(0, 0)$ pairs of (a, a'). Hence the total number of $(0, 0)$ pairs and $(1, 1)$ pairs of (a, a') is bounded above by the total number of $(0, 1)$ pairs of (b, b') which is $r = 1$, giving $n \leq 2$, a contradiction.

It is now known that (b, b') has at least one $(1, 1)$ pair, and $n > 2$ and $r > 0$ can be assumed; without loss of generality, let $(b_j, b'_j) = (1, 1)$. Let (e, e') be the boundary of (b, b') created by changing a $(0, 1)$ pair to a $(0, 0)$ pair at the kth coordinate. Then define (c, c') from (e, e') by changing the $(1, 1)$ pair to a $(0, 1)$ pair at the jth coordinate.

$$\begin{array}{ccc}
\begin{pmatrix} b \\ b' \end{pmatrix} & = & \begin{pmatrix} \ldots 1 \ldots 0 \ldots \\ \ldots 1 \ldots 1 \ldots \end{pmatrix} \\
\begin{pmatrix} e \\ e' \end{pmatrix} & = & \begin{pmatrix} \ldots 1 \ldots 0 \ldots \\ \ldots 1 \ldots 0 \ldots \end{pmatrix} \\
\begin{pmatrix} c \\ c' \end{pmatrix} & = & \begin{pmatrix} \ldots 0 \ldots 0 \ldots \\ \ldots 1 \ldots 0 \ldots \end{pmatrix} \\
 & & \ldots j \ldots k \ldots
\end{array}$$

Now (e, e') is also in the boundary of (c, c'), which is smaller than (b, b'); hence (e, e') is not added to $\partial(S_r^q - \{(a, a')\})$ when (b, b') is added to $S_r^q - \{(a, a')\}$. Since such a boundary element is created for each $(0, 1)$ pair of (b, b'), at most r boundary elements are added to $\partial(S_r^q - \{(a, a')\})$ when (b, b') is added to $S_r^q - \{(a, a')\}$.

At least r $(r-1)$-faces are removed from ∂S_r^q when (a, a') is taken away from S_r^q. At most r $(r-1)$-faces are added to $\partial(S_r^q - \{(a, a')\})$ when (b, b') is added to $S_r^q - \{(a, a')\}$. The desired result is achieved; i.e., $|\partial((S_r^q - \{(a, a')\}) \cup \{(b, b')\})| \leq |\partial S_r^q|$. To further show that $|\partial RS_r| \leq |\partial R_s^q|$ is a straightforward matter of iteration, as Lindström discusses in [4]. Let $T = (S_r^q - \{(a, a')\}) \cup \{(b, b')\}$. If $T \neq RS_r$, let (a, a') be the last r-face in T and let (b, b') be the first r-face not in T. T must satisfy the same properties as S_r^q after the sequential replacement step. If not, the sequential replacement step is able to replace faces of T. Let the first such replacement be from (x, x') to (y, y'), where $(x, x') \in T$ and $(y, y') \notin T$. Since $(b, b') < (y, y') < (x, x') < (a, a')$, $(x, x') \in S_r^q$ and $(y, y') \notin S_r^q$. Then the sequential replacement step would also have replaced (x, x') to (y, y') in S_r^q. Hence $R_l T = T$ for all $l \in \{1, \ldots, n\}$, and the process of arbitrary replacement may be repeated with T in the place of S_r^q until the set RS_r is obtained. The final result below is obtained.

$$|\partial RS_r| \leq |\partial S_r|$$

Now $R\partial S_r$ is an initial set with respect to the total order on $(r-1)$-faces, so if ∂RS_r is also initial with respect to this order, it is clear that $\partial RS_r \subseteq R\partial S_r$, which is precisely the statement of Theorem 1, the crucial inclusion. By the following lemma (proved in [4]), ∂RS_r is indeed initial, so Theorem 1 holds.

Lemma 1. *If $RS_r = S_r$, then $R\partial S_r = \partial S_r$.*

The proof of this lemma is included in the Appendix.

3.3 Proof of the Corollary to Theorem 1

Proof of Corollary 1. To prove the first inequality of the corollary, $|\partial^p RS_r| \leq |\partial^p S_r|$, it is sufficient to show

$$\partial^p RS_r \subseteq R\partial^p S_r, \quad p = 1, \ldots, r. \tag{6}$$

The case for $p = 1$ is the statement of Theorem 1. By induction, suppose $\partial^{p-1} RS_r \subset R\partial^{p-1} S_r$ holds. Since both sets consist of faces in the n-cube, applying ∂ to both sides yields

$$\partial^p RS_r \subseteq \partial R\partial^{p-1} S_r, \tag{7}$$

and applying Theorem 1 to the right side of (7) yields the desired result in (6). Since $|R\partial^p S_r| = |\partial^p S_r|$, the first inequality of Corollary 1 follows.

To prove the second inequality of the corollary, $|\partial^{-p} S_r| \leq |\partial^{-p} RS_r|$, it is sufficient to show by induction

$$R\partial^{-p} S_r \subseteq \partial^{-p} RS_r, \quad p = 1, \ldots, n - r.$$

For the case $p = 1$, recall that by definition of the boundary operator

$$\partial^{-1}\partial S_r = S_r, \quad \partial\partial^{-1} S_r \subseteq S_r. \tag{8}$$

If S_r is replaced by $\partial^{-1} S_r$ in the theorem, applying (8) and then ∂^{-1} yields

$$\begin{aligned} \partial R\partial^{-1} S_r &\subseteq R\partial\partial^{-1} S_r \subseteq RS_r, \\ R\partial^{-1} S_r &\subseteq \partial^{-1} RS_r. \end{aligned} \tag{9}$$

By induction, suppose $R\partial^{-(p-1)} S_r \subseteq \partial^{-(p-1)} RS_r$ holds. Then replacing S_r with $\partial^{-(p-1)} S_r$ in (9) and applying the induction for $(p - 1)$ yields

$$R\partial^{-p} S_r = R\partial^{-1}\partial^{-(p-1)} S_r \subseteq \partial^{-1} R\partial^{-(p-1)} S_r \subseteq \partial^{-1}\partial^{-(p-1)} RS_r = \partial^{-p} RS_r.$$

The second inequality of Corollary 1 follows from this inclusion, which completes the proof.

Example 7. Consider the set S_2 of 2-faces which lie in the 3-cube, as shown in Figure 7(a). S_2 consists of the faces w_{03} and w_{47}. The second boundary of S_2, $\partial^2 S_2$, is shown in Figure 7(b); i. e. , all eight vertices of the 3-cube. After the replacement operator R acts on S_2, the result appears as RS_2, in Figure 7(c). The number of vertices in the second repeated boundary of RS_2, however, is only six, as shown in Figure 7(d). This agrees with the Corollary to Theorem 1, which states for $p = 2$ that $|\partial^2 RS_2| \leq |\partial^2 S_2|$.

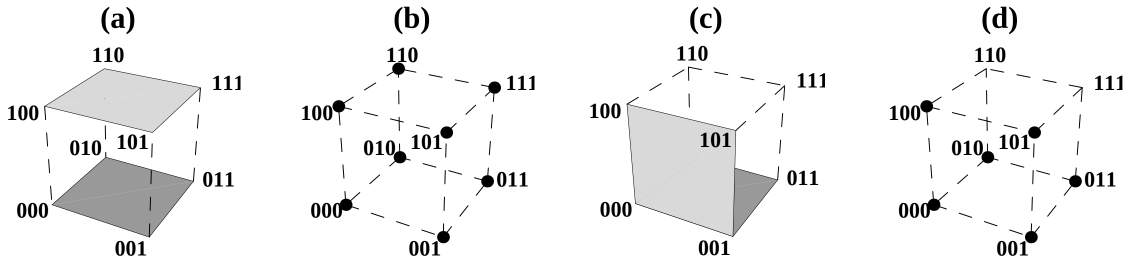

Figure 7: A set of 2-faces, S_2, appears in (a), and the vertices of the boundary $\partial^2 S_2$ appear in (b). After the replacement operator is applied to S_r, RS_r is depicted in (c), with its smaller number of boundary vertices $\partial^2 RS_r$ in (d).

Now it has been shown that the minimum(maximum) number of faces in the boundary(coboundary) of the set S_r of r-faces of a cubical complex C is bounded below(above) by that in the case where S_r is an initial set of r-faces, further attention is restricted to this case, $S_r = RS_r$. After reviewing the derivation of the formula for the optimal number of faces in the boundary or coboundary of a set of r-faces of a simplicial complex, the formula for the exact number of faces in the boundary or coboundary of an initial set of r-faces will be obtained.

4 The Optimal Number of Faces for Simplicial Complexes

The derivation of a closed formula for the optimal number of faces in cubical complexes requires a previous result for the optimal number of faces in simplicial complexes shown by J. B. Kruskal in [3]. What follows here is a summary of the necessary results from Kruskal's work in order to derive the formula for cubical complexes.

Definition. *A simplicial (or binary) complex Δ is a finite collection of finite sets or faces which satisfy the property that if $F \in \Delta$ is a set of the simplicial complex, then all subsets $G \subseteq F$ must also be in Δ.*

A set of a simplicial complex is also referred to as a face. A standard representation of simplicial complexes is obtained by representing the faces as binary words of length n; subfaces are determined by binary inclusion, with the empty face represented as the binary word with n 0's. The *weight* of a binary word is the number of 1's that appear in the word. A total order is established on the faces by inheriting the total order on the binary words.

Example 8. Simplicial complexes are best viewed as vertices, edges, triangles, tetrahedrons, and higher dimensional structures. Cubical complexes are divided into those that can be embedded in n-cubes and those that cannot, but all simplicial complexes can be viewed as a finite, arbitrary collection of vertices in Euclidean space $\Re^n$ with the structure of edges, triangles, etc., superimposed. A face of a simplicial complex is denoted by the vertices that appear in it. The simplicial complex, Δ, in Figure 8 consists of the triangle ABC, the edge BD, the tetrahedron $DEFG$, and all subfaces of these three faces. The complex can be written in terms of its vertices: $\Delta = \{\emptyset, A, B, C, D, E, F, G, AB, AC, BC, BD, DE, DF, DG, EF, EG, FG, ABC, DEF, DEG, DFG, EFG, DEFG\}$. If the complex is written in terms of the faces that generate it, then $\Delta = \langle DEFG, ABC, BD\rangle$. To convert the simplicial complex into a binary complex, represent A through G as the least significant bit through the most significant bit, respectively, of a 7-bit binary word; i. e. , $A = 0000001$, $B = 0000010$, and so forth. Then in terms of binary word generators, the binary complex can be written as $\Delta = \langle 1111000, 0000111, 0001010 \rangle$.

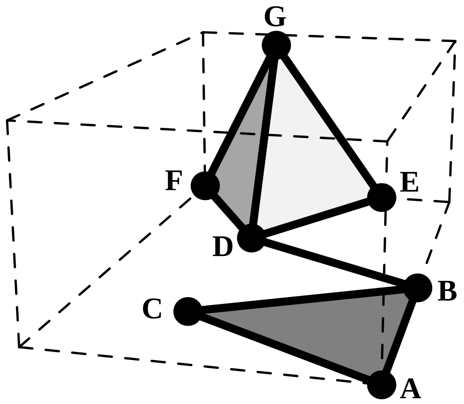

Figure 8: The simplicial complex consists of the tetrahedron $DEFG$, the triangle ABC, the edge BD, and all subfaces of these faces.

Before the results are stated, some notation must be defined. As in [3] for any positive integer m define the *r-canonical representation*

$$m = \binom{m_1}{r} + \binom{m_2}{r-1} + \ldots + \binom{m_k}{r-k+1},$$

where m_i for $1 \leq i \leq k$ is chosen as large as possible such that

$$\binom{m_1}{r} + \ldots + \binom{m_i}{r-i+1} \leq m$$

until equality is obtained at $i = k$. Kruskal shows that this representation is unique and that the following relationships hold:

$$m_1 > m_2 > \ldots > m_k \geq r - k + 1 \geq 1.$$

Define for any sequence of positive integers $m_1, \ldots, m_k$

$$[m_1, \ldots, m_k]_r = \binom{m_1}{r} + \binom{m_2}{r-1} + \ldots + \binom{m_k}{r-k+1}.$$

Then if $m = [m_1, \ldots, m_k]_r$ is the r-canonical representation of m, Kruskal defines the *fractional pseudopower* $m^{(s/r)}$ to be

$$m^{(s/r)} = [m_1, \ldots, m_k]_s,$$

with $0^{(s/r)} = 0$. The background is complete for the next theorem.

Kruskal's Theorem. *If a simplicial complex Δ has exactly m words of weight r, and if $s > r (s < r)$, then the maximum(minimum) number of integers of weight s that Δ can have is $m^{(s/r)}$.*

This theorem provides the optimization formula for simlicial complexes and is analogous to the optimization formula for cubical complexes which will be derived in the next section. The proof of this theorem, which appears in [3], follows with some work by the application of the next two lemmas.

Lemma 2. *Let $m_1, \ldots m_k$ be nonnegative integers. Then the number of integers of weight r, which are less than $2^{m_1} + \ldots + 2^{m_k}$, is $[m_1, \ldots, m_k]_r$.*

If S is a set of binary words of weight r, define the set $r(S)$ of *related words* to be the set of all words x such that either $x \subseteq y$ for some $y \in S$ or $y \in S$ for each $y \subset x$. This provides the setting for the next lemma.

Lemma 3. *Let S be the m first nonnegative integers of weight r. Then the number of integers of weight s in $r(S)$ is $m^{(s/r)}$, and $r(S)$ is the set of all non-negative integers which are less than $2^{m_1} + \ldots + 2^{m_k}$, if $m = [m_1, \ldots, m_k]_r$ is the r-canonical representation of m.*

The proof of Lemmas 2 and 3 are given in Appendix A; these proofs demonstrate tools that are used to prove the analogous computational result for cubical complexes. Lemmas 2 and 3 lead to a proof of Kruskal's theorem, but this proof is omitted as it has less relevance to the proof of the analogous computational result for cubical complexes.

5 The Optimal Number of Faces for Cubical Complexes

As discussed in the previous section, the definition of the fractional pseudopower is crucial to the development of a computational formula for the optimal number of faces in a simplicial complex. This definition allows the counting of the number of binary words of weight r that are less than a fixed binary word. For cubical complexes, it is useful to define a similar formula to count the number of faces of dimension r which are less than a fixed vertex (recalling that a vertex is a 0-dimensional face). To this end, the following function defined by Kruskal in [2] is required:

$$g(r,i,m) = \sum_{v=0}^{r} \binom{m}{r-v} \binom{i-1}{v} 2^{m-(r-v)}. \tag{10}$$

Kruskal used this function in an attempt at defining the optimality conditions for cubical complexes, which are correct for a relatively small number of cases. Lindström demonstrates in [4] that (10) is also given as

$$g(r,i,m) = \sum_{v=0}^{m} \binom{m}{v} \binom{v+i-1}{r}.$$

Lindström takes this function and develops the corresponding pseudopower expression for cubical complexes. Analogously to the result for fractional pseudopowers arising from Pascal's triangle,

$$g(r,i,m) + g(r,i+1,m) = g(r,i,m+1). \tag{11}$$

Given an integer $m > 0$, determine $m_1, m_2, \ldots, m_k$ such that

$$m \geq g(r,1,m_1) + \ldots + g(r,k,m_k) = m_{(r/r)}, \tag{12}$$

where first m_1 is chosen to be as large as possible such that $m \geq g(r,1,m_1)$, then m_2 is chosen to be as large as possible such that $m \geq g(r,1,m_1) + g(r,2,m_2)$, and so forth. Equality is not always obtained, but the resulting sum in (12) is defined to be $m_{(r/r)}$, and it follows from (11) that

$$m_1 > m_2 > \ldots > m_k \geq r - k + 1. \tag{13}$$

For the sum defined in (12) for $m > 0$, define the *lower semipower*, $m_{(s/r)}$, by

$$m_{(s/r)} = g(s, 1, m_1) + \ldots + g(s, k, m_k). \tag{14}$$

The lower semipower is defined in anology with Kruskal's fractional pseudopower, which will also be referred to as the *upper semipower*. Now Lindström's main result, which appears as Theorem 3 in [4], can be stated as a computational formula for the optimal number of faces in cubical complexes involving upper and lower semipowers.

Theorem 2. *If a cubical complex C has m faces of dimension r, and if $s > r (s < r)$, then the maximum(minimum) number of faces of dimension s that it can have is*

$$m_{(s/r)} + (m - m_{(r/r)})^{(s/r)}.$$

Theorem 2 follows from two lemmas which appear as Lemma 4 and Lemma 5 in [4], which Lindström gives without proof. Lemmas 4 and 5 are analogous to Lemmas 2 and 3 for simplicial complexes, and their proofs are similar. These lemmas are stated below with proof, and afterward the proof of Theorem 2 is given (of which Lindström gives a brief version in [4]).

Lemma 4. *Let $m_1 > m_2 > \ldots > m_k$ be nonnegative integers. Then the number of r-faces w_{ij}, with $j < 2^{m_1} + \ldots + 2^{m_k}$, is*

$$g(r, 1, m_1) + \ldots + g(r, k, m_k).$$

Proof of Lemma 4. It is convenient to first partition S into k disjoint sets that are each counted by the function given in (10). Let S be the set of r-faces w_{ij} with

Table 1: Some Computational Results for Theorem 2.

m	r	s	$m_{(s/r)} + (m - m_{(r/r)})^{(s/r)}$	m	r	s	$m_{(s/r)} + (m - m_{(r/r)})^{(s/r)}$
0	0	1	0	0	3	2	0
1	0	1	0	1	3	2	6
2	0	1	1	2	3	2	11
4	0	1	4	4	3	2	18
8	0	1	12	0	3	1	0
16	0	1	32	1	3	1	12
0	1	2	0	2	3	1	20
1	1	2	0	4	3	1	28
2	1	2	0	0	3	0	0
4	1	2	1	1	3	0	8
8	1	2	2	2	3	0	12
16	1	2	7	4	3	0	15

$j < 2^{m_1} + \ldots + 2^{m_k}$. For $l \in \{1, \ldots k\}$, define S_l to be the set of r-faces w_{ij} of S for which j has a 1 in the $2^{m_1}, \ldots, 2^{m_{l-1}}$ positions and a 0 in the 2^{m_l} position. It can be shown that

$$S = S_1 \cup \ldots \cup S_k, \tag{15}$$

and furthermore that the right-hand side of (15) is a disjoint union.

To show containment in the forward direction, let w_{ij} be an r-face of S. Then $j < 2^{m_1} + \ldots + 2^{m_k}$; hence j must differ from $2^{m_1} + \ldots + 2^{m_k}$ at some first position 2^{m_l}, and $j \in S_l$ by definition. It is easy to see that $l \in \{1, \ldots, k\}$. The reverse containment follows from the definition of the sets $S_1, \ldots, S_k$. To show that the right-hand side of (15) is a disjoint union, let $p, q \in \{1, \ldots, k\}$. Without loss of generality suppose $p > q$. Let w_{ij} and w_{uv} be members of S_p and S_q, respectively. Then j has a 1 at the 2^{m_q} position and v has a 0 at the 2^{m_q} position; hence $w_{ij} \neq w_{uv}$ and $S_p \cap S_q = \emptyset$ for all $p \neq q$.

Now fix $l \in \{1, \ldots, k\}$ and consider $|S_l|$. Let w_{ij} be an arbitrary member of S_l. Then j has 1's in the $l - 1$ positions $2^{m_1}, \ldots, 2^{m_{l-1}}$, a 0 in the 2^{m_l} position, and either a 1 or a 0 in each of m_l positions to the right of the 2^{m_l} position. Thus w_{ij}

can have a $(0,1)$ or a $(1,1)$ pair in each of the $m_1, \ldots, m_{l-1}$ coordinates, $(0,0)$ pairs in the m_l coordinate and all the other coordinates to the left of the m_l coordinate, and a $(0,0)$, $(0,1)$ or a $(1,1)$ pair in each of the coordinates to the right of the m_l coordinate. Since the dimension of w_{ij} is r, w_{ij} must have r $(0,1)$ pairs; let v be the number of $(0,1)$ pairs of w_{ij} which occur to the left of the m_l coordinate. There are $l-1$ coordinates in which to place these v pairs, leaving the remaining coordinates fixed at $(1,1)$. Since $r-v$ of the $(0,1)$ pairs occur to the right of the m_l coordinate, there are m_l positions in which to place these $r-v$ pairs, leaving a choice between $(0,0)$ and $(1,1)$ for the remaining $m_l-(r-v)$ coordinates. Since v may range from 0 to r, the following formula for $|S_l|$ arises:

$$|S_l| = \sum_{v=0}^{r} \binom{m_l}{r-v} \binom{l-1}{v} 2^{m_l-(r-v)} = g(r,l,m_l). \tag{16}$$

The result follows by letting l range from 1 to k, completing the proof of Lemma 4.

As in the analogous proof for simplicial complexes, the background for the next lemma is set by defining, for a set S of r-faces, $r(S)$ to be the set of faces related to S; that is, $r(S)$ is the set of faces w_{uv} such that w_{uv} is either a subface of some r-face w_{ij} in S, or all the r-faces w_{ij} which are subfaces of w_{uv} are contained in S. This definition can also be written as

$$r(S) = \bigcup_{p=-\infty}^{r} \partial^p(S).$$

Note that when $p > r$ that $\partial^p(S) = \emptyset$. Also, r(S) is a finite set since a single face of dimension $s > r$ contains 2^{s-r} subfaces of dimension r; in other words, there exists some positive integer B such that

$$\bigcup_{p=-\infty}^{r} \partial^p(S) = \bigcup_{p=-B}^{r} \partial^p(S).$$

Lemma 5. *Assume that $m = m_{(r/r)}$ and let S be the m first r-cubes w_{ij} in the total ordering of cubes. Then the number of s-faces of $r(S)$ is $m_{(s/r)}$, and $r(S)$ is the set* *of all faces* w_{uv} *such that* $v < 2^{m_1} + \ldots + 2^{m_k}$*, if* $m \geq g(r,1,m_1) + \ldots + g(r,k,m_k) = m_{(r/r)}$ *under the conditions of (13).*

Proof of Lemma 5. Analogously to the proof for simplicial complexes, it is useful to first prove the following set equality:

$$r(S_r) = \{w_{uv} : v < 2^{m_1} + \ldots + 2^{m_k}\}. \tag{17}$$

For the forward containment of (17), let $w_{uv} \in r(S_r)$, and let s be the dimension of w_{uv}. Two cases arise.

1. If $s \leq r$, then w_{ij} is in the $(r-s)$th boundary of an r-face $w_{ij} \in S_r$, and w_{uv} is created from w_{ij} by changing $(0,1)$ pairs to $(0,0)$ or $(1,1)$ pairs; in any event, $v \leq j$, and since $j < 2^{m_1} + \ldots + 2^{m_k}$, containment holds.

2. if $s > r$, then S_r must contain the 2^{s-r} r-faces in the $(s-r)$th coboundary of w_{uv}. Suppose to the contrary that $v \geq 2^{m_1} + \ldots + 2^{m_k}$. Then by changing $(s-r)$ of the $(0,1)$ pairs of w_{uv} to $(1,1)$ pairs, the r-face w_{iv} for some binary word i is formed; by assumption this r-face must be in S_r, but $v \geq 2^{m_1} + \ldots + 2^{m_k}$ contradicts the construction of S_r.

In either case, with Lemma 4 the forward containment follows.

For the reverse containment of (17), let w_{uv} be an s-face which satisfies $v < 2^{m_1} + \ldots + 2^{m_k}$. Two cases arise.

1. If $s \leq r$, let m_l be the first coordinate of w_{uv} for which v has a 0; thus v has 1's in the $2^{m_1}, \ldots 2^{m_{l-1}}$ positions. By (13), there are at least $r - l + 1$ positions after the 2^{m_l} position, or at least r total coordinates where either a $(0,1)$ pair exists in w_{uv}, or a $(0,0)$ or $(1,1)$ pair of w_{uv} can be changed to $(0,1)$ to construct an r-face w_{ij} such that $w_{ij} \in \partial^{s-r}(\{w_{ij}\})$ and $j < 2^{m_1} + \ldots + 2^{m_k}$. By (16) and Lemma 4, the containment holds.

2. If $s > r$, It is sufficient to show that $\partial^{s-r}(\{w_{uv}\}) \subseteq S_r$. The $(s-r)$th boundary of w_{uv} is created by choosing $(s-r)$ of the $(0,1)$ pairs of w_{uv} and constructing the r-face w_{ij} changing these pairs individually to either $(0,0)$ or $(1,1)$. In any event, $j \le v$; by Lemma 4 $w_{ij} \in S$, and by (16) the containment holds.

The equality in (17) follows from these containments.

From the previous equality, the number of s-faces in $r(S)$ is the same as the number of s faces w_{uv} which satisfy $v < 2^{m_1} + \ldots + 2^{m_k}$. By Lemma 4, this is just $g(s,1,m_1) + \ldots + g(s,k,m_k)$; but this is $m_{(s/r)}$ by (14). This completes the proof of Lemma 5.

Proof of Theorem 2. Let S_r be the r-faces of the cubical complex C, with $m = |S_r|$. Corollary 1 to Theorem 1 allows the assumption that S_r is an initial set of r-faces. If $s > r$, the set S_r has at most as many s-faces in its $(s-r)$th coboundary as does the set RS_r; hence the possible number of s-faces in C is maximized when $S_r = RS_r$. If $s < r$, the set S_r has at least as many s-faces in its $(r-s)$th boundary as does the set RS_r; hence the possible number of s-faces in C is minimized when $S_r = RS_r$.

If $m = m_{(r/r)}$, Theorem 2 follows from Lemmas 4 and 5. If $s < r$, by definition of cubical complexes, C contains all of the faces of $r(S_r)$ that have dimension less than r. If $s > r$, C contains a subset (possibly all) of the s-faces of $r(S_r)$. In either case, the optimal number of s-faces in C is $m_{(s/r)}$.

Next assume that $m > m_{(r/r)} = g(r,1,m_1) + \ldots + g(r,k,m_k)$ and let $M = 2^{m_1} + \ldots + 2^{m_k}$. Then S_r contains $m_{(r/r)}$ faces w_{ij} with $j < M$ and $m - m_{(r/r)}$ faces w_{iM}, by Lemma 4. Let w_{uM} be an arbitrary s-face; $w_{uM} \in r(S_r)$ if and only if w_{uM} is a subface of some r-face $w_{iM} \in S_r$ or every r-face w_{iM} in the boundary of w_{uM} is in S_r. This follows by the definition of $r(S_r)$ and the observation that if $s > r$, all the r-subfaces w_{ij} for which $j \neq M$ satisfy $j < M$ and $w_{ij} \in S_r$.

Since M is fixed, it is useful to look at the complements $M - u$ and $M - i$ for an s-face and an r-face, respectively. Note that $M - u$ is a binary word of weight

s, since w_{uM} has s $(0,1)$ pairs. Similarly, $M-i$ is a binary word of weight r. If $s>r$, an r-face w_{iM} is contained in the boundary of an s-face w_{uM} if and only if $M-i \subset M-u$. Couching the previous bijection in terms of binary words, $w_{uM} \in r(S_r)$ if and only if $M-u \subseteq M-i$ for some r-face w_{iM}, or every r-face w_{iM} is in S_r when $M-i \subset M-u$. By the total ordering on cubical faces, the $m-m_{(r/r)}$ faces w_{iM} in S_r correspond to the $m-m_{(r/r)}$ first binary words $M-i$ of weight r. By Lemma 3, the optimal number of binary words of weight s that a binary complex with the $m-m_{(r/r)}$ first binary words of weight r can have is $(m-m_{(r/r)})^{(s/r)}$. By the bijection above, these binary words of weight s correspond to the $(m-m_{(r/r)})^{(s/r)}$ s-faces w_{uM} in $r(S_r)$. Thus the set $r(S_r)$ consists of $m_{(s/r)}$ s-faces w_{uv} with $v<2^{m_1}+\ldots+2^{m_k}$ and $(m-m_{(r/r)})^{(s/r)}$ s-faces w_{uM}. This completes the proof of Theorem 2.

6 Characterization of f-vectors of Cubical Complexes

Theorem 2 leads to a characterization of cubical complexes in terms of the allowable number of faces of a given dimension. If a cubical complex C has m faces of dimension $r<s$, then it cannot have more than $m_{(s/r)}+(m-m_{(r/r)})^{(s/r)}$ faces of dimension s; of course a similar statement can be made for the case $r>s$. If the information about the number of faces of a given dimension of a candidate cubical complex C is available, application of the theorem determines whether or not C is a legal cubical complex. This information about the cubical complex is called the *f-vector* of the complex.

Definition. *Let C be a cubical complex and let $f=(f_0,f_1,\ldots,f_d)$ be a $(d+1)$-tuple of positive integers. Then f is the f-vector of C if and only if C has f_i faces of dimension i (the f-vector of the empty complex is the empty tuple). Furthermore, the dimension of the complex is d (or -1 for the empty complex).*

The f-vector is the necessary tool for storing the information about the number of faces of each dimension of the cubical complex. For example, Theorem 2 can be applied to an f-vector f of a cubical complex C, by choosing two nonnegative integers $0 \leq i < j \leq d$, and verifying $f_j \leq (f_i)_{(s/r)} + (f_i - (f_i)_{(r/r)})^{(s/r)}$. This leads to the next theorem.

Theorem 3. *Let $f = (f_0, f_1, \ldots, f_d)$ be a $(d+1)$-tuple of positive integers. Then f is the f-vector of a cubical complex C if and only if*

$$f_{i+1} \leq (f_i)_{(i+1/i)} + (f_i - (f_i)_{(i/i)})^{(i+1/i)}, \quad i = 0, \ldots, d-1. \tag{18}$$

Table 2: Some Computational Results for Theorem 3.

f-vector	corresponding complex?	notes
(4,3,1)	no	a square must have four sides
(7,9,3)	yes	confirms example in Introduction
(7,10,3)	no	$f_1 = 10$ is too large given $f_0 = 7$
(9,14,7)	no	$f_1 = 14$ is too large given $f_0 = 9$
(8,12,6,1)	yes	the 3-cube
(9,14,7)	no	$f_1 = 14$ is too large given $f_0 = 9$; $f_1 = 14$ is too small given $f_2 = 7$

Proof of Theorem 3. The forward direction follows by repeated application of Theorem 2. For the reverse direction, let $f = (f_0, f_1, \ldots, f_d)$ be a $(d+1)$-tuple of positive integers which satisfies (18). It can be shown by construction that there exists a cubical complex C that has f as an f-vector. Let S_i be the first f_i i-faces in the total ordering, for $i = 0, \ldots, d$. Initialize the construction of C by letting $C = S_0$, the first f_0 0-faces (vertices) of the total ordering. As seen in the proof of Theorem 2, $r(S_0)$ contains the $((f_0)_{(1/0)} + (f_0 - (f_0)_{(1/0)})^{(1/0)})$ first 1-faces in the total order. By assumption, this is an upper bound for f_1; by letting $C{=}C \cup S_1$, C is still a cubical complex since $S_1 \subseteq r(S_0)$ (the boundary of S_1 is also in C). Continue this process similarly for each $0 \leq i \leq d$ to obtain the cubical complex $C = S_1 \cup \ldots \cup S_d$ for which $\partial S_d \subseteq S_{d-1}, \ldots, \partial S_1 \subseteq S_0$. This completes the proof of Theorem 3.

7 f-Vectors of Arbitrary Cubical Complexes

The proofs of Theorems 2 and 3 completely ignore the case of cubical complexes which do not lie in cubes. The examples here show that Theorem 3 does not hold for arbitrary cubical complexes, leaving open the question of characterization of f-vectors of arbitrary cubical complexes.

Definition. *A cubical complex C (not necessarily embeddable in a cube) is a finite collection of finite sets which satisfy the following properties:*

1. If $F \in C$ and G is a subset of F, then $G \in C$.

2. If F is a member of C, then F is a cube of finite dimension.

A cubical complex that contains a triangle or some other odd-sided polygon has a closed path of odd length and is not embeddable in a cube (recall that closed paths in cubes must have even length). For similar reasons, a cubical complex which contains three squares forming a triangle, i. e. , the cross product of a triangle and an edge, is not embeddable in a cube. Furthermore, Theorem 3 does not hold for these complexes, as is illustrated in the final example.

Example 9. The cubical complex C in Figure 9 can be viewed as the cross product of a triangle and a line and has the associated f-vector $(6, 9, 3)$. Theorem 3 fails for this f-vector in two places. According to the theorem C must have at least 7 0-faces to have 9 1-faces, and C can have at most 2 2-faces given that it has 6 0-faces.

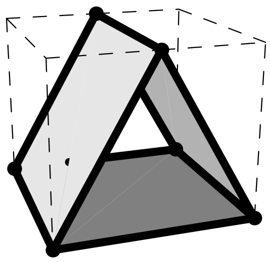

Figure 9: Theorem 3 fails for this cubical complex.

A Appendix: Proofs of Technical Lemmas

Lemma 1. *If $RS_r = S_r$, then $R\partial S_r = \partial S_r$.*

Proof of Lemma 1. It must be shown that if S_r is an initial set of r-faces, then the boundary set ∂S_r must be an initial set of $(r-1)$ faces. Let $(f, f') \in \partial S_r$ be an $(r-1)$-face. To prove the lemma it is sufficient to show that whenever (e, e') is an $(r-1)$-face and $(e, e') < (f, f')$, then $(e, e') \in \partial S_r$.

If $(f, f') \in \partial S_r$, then there must be an r-face $(h, h') \in S_r$ such that $(f, f') \in \partial\{(h, h')\}$. From this point there are two cases, $e' = h'$ and $e' < h'$.

Case 1. Assume $e' = h'$. For $n < 3$ there are a small number of cases which are easily verified. The total order on faces forces $e' = f' = h'$. In this case, (f, f') cannot have been created from (h, h') by changing a $(0, 1)$ pair to a $(0, 0)$ pair, since this would violate $f' = h'$. Hence $f = h$ except for a single index j for which $f_j = 1$ and $h_j = 0$. The total order on faces requires $e > f$, so let k be the index for which $e_i = f_i$ for all indices i less than k and $e_k > f_k$. Note that $e'_k = f'_k = 1$ since $e' = f'$. Also, $j \neq k$ since $f_j = 1$ and $f_k = 0$. If $j < k$ the situation below is obtained.

$$\begin{array}{ccc}
\begin{pmatrix} h \\ h' \end{pmatrix} & = & \begin{pmatrix} \ldots 0 \ldots 0 \ldots \\ \ldots 1 \ldots 1 \ldots \end{pmatrix} \\
\begin{pmatrix} f \\ f' \end{pmatrix} & = & \begin{pmatrix} \ldots 1 \ldots 0 \ldots \\ \ldots 1 \ldots 1 \ldots \end{pmatrix} \\
\begin{pmatrix} e \\ e' \end{pmatrix} & = & \begin{pmatrix} \ldots 1 \ldots 1 \ldots \\ \ldots 1 \ldots 1 \ldots \end{pmatrix} \\
 & & \ldots j \ldots k \ldots
\end{array}$$

Construct the r-face (g, g') by letting $(g_i, g'_i) = (e_i, e'_i)$ for all indices i except for $i = j$, for which $(g_j, g'_j) = (0, 1)$.

$$\begin{array}{ccc}
\begin{pmatrix} g \\ g' \end{pmatrix} & = & \begin{pmatrix} \ldots 0 \ldots 1 \ldots \\ \ldots 1 \ldots 1 \ldots \end{pmatrix} \\
 & & \ldots j \ldots k \ldots
\end{array}$$

This gives $g > h$ and $(g, g') < (h, h')$. But if S_r is an initial set of r-faces and $(h, h') \in S_r$, certainly $(g, g') \in S_r$, and since $(e, e') \in \partial\{(g, g')\}$, by definition $(e, e') \in \partial S_r$.

If $j > k$ the situation below is obtained.

$$\begin{array}{ccc}
\begin{pmatrix} h \\ h' \end{pmatrix} & = & \begin{pmatrix} \ldots 0 \ldots 0 \ldots \\ \ldots 1 \ldots 1 \ldots \end{pmatrix} \\
\begin{pmatrix} f \\ f' \end{pmatrix} & = & \begin{pmatrix} \ldots 0 \ldots 1 \ldots \\ \ldots 1 \ldots 1 \ldots \end{pmatrix} \\
\begin{pmatrix} e \\ e' \end{pmatrix} & = & \begin{pmatrix} \ldots 1 \ldots e_j \ldots \\ \ldots 1 \ldots 1 \ldots \end{pmatrix} \\
 & & \ldots k \ldots j \ldots
\end{array}$$

If there is an index $l > k$ for which $e_l = 1$, construct (g, g') by letting $(g_i, g'_i) = (e_i, e'_i)$ for all indices i except $i = l$ and let $(g_l, g'_l) = (0, 1)$. Then $g > h$, hence $(g, g') < (h, h') \Rightarrow (g, g') \in S_r$. Thus $(e, e') \in \partial S_r$.

If for all indices $i > k$ we have $e_i = 0$, it is necessary to see that e and f have the same number of 1's. This follows from seeing that e and f have the same number of 0's. Both (e, e') and (f, f') are $(r-1)$-faces, and $e' = f'$. Let a be the number of 0's in e' (or f'). Then both e and f have 0's in the corresponding coordinates since $(1, 0)$ is not a legal pair. The only other 0's of e or f match with 1's in e' or f', respectively, giving $r - 1$ such $(0, 1)$ pairs in (e, e') (or (f, f')) and $a + r - 1$ total 0's in e (or f). Thus e and f have the same number of 0's and the same number of 1's.

Since $e_i = f_i$ holds for all $1 \leq i < k$, and $e_i = 0$ holds for all $i > k$, as e and f have the same number of 1's $h_i = f_i = e_i = 0$ is obtained for all $i \in \{k+1, \ldots, j-1, j+1, \ldots, n\}$. It is easy to see now that $(e, e') \in \partial\{(h, h')\}$ by changing the kth coordinate of (h, h') from $(0, 1)$ to $(1, 1)$; hence $(e, e') \in \partial S_r$.

Case 2. Assume that $e' < h'$. If there is an index j such that $(e_j, e'_j) = (1, 1)$, construct (g, g') by letting $(g_i, g'_i) = (e_i, e'_i)$ for all indices i except $i = j$, and let $(g_j, g'_j) = (0, 1)$. Since $g' = e' < h'$, $(g, g') < (h, h')$ and $(e, e') \in \partial\{(g, g')\}$ give $(e, e') \in \partial S_r$, because S_r is an initial set of r-faces which contains $(h, h') > (g, g')$.

Otherwise, suppose that $(e_i, e'_i) \in \{(0, 1), (0, 0)\}$ for all indices i. Since $e' < h'$, let j be the index for which $e'_i = h'_i$ for all indices $i < j$ and $e'_j < h'_j$. If $(e_k, e'_k) = (0, 0)$

for some $k > j$, the situation below arises.

$$\begin{array}{ccc} \begin{pmatrix} h \\ h' \end{pmatrix} & = & \begin{pmatrix} \ldots h_j \ldots h_k \ldots \\ \ldots 1 \ldots h'_k \ldots \end{pmatrix} \\ \begin{pmatrix} e \\ e' \end{pmatrix} & = & \begin{pmatrix} \ldots 0 \ldots 0 \ldots \\ \ldots 0 \ldots 0 \ldots \end{pmatrix} \\ & & \ldots j \ldots k \ldots \end{array}$$

Let $(g_i, g'_i) = (e_i, e'_i)$ for all indices i except $i = k$, and let $(g_k, g'_k) = (0, 1)$. Then $g' < h'$ and $(g, g') < (h, h')$; hence, $(e, e') \in \partial S_r$.

Otherwise, $(e_i, e'_i) = (0, 1)$ for all indices $i > j$. Then

$$(e, e') = (0 \ldots 0, h'_1 \ldots h'_{j-1} 0 1 \ldots 1).$$

The face (e, e') has dimension $r - 1$ and hence $r - 1$ coordinates $(0, 1)$. For each index $i > j$, $(e_i, e'_i) = (0, 1)$, giving $n - j$ of these $(0, 1)$ coordinates. The remainder must come from $(r - 1) - (n - j)$ of the coordinates (e_i, e'_i) where $1 \leq i < j$. Thus h' has $h'_i = 1$ for $(r - 1) - (n - j)$ of the indices $1 \leq i < j$ and $h'_i = 0$ for the rest of those indices. But (h, h') has dimension r, and must satisfy $(h_i, h'_i) = (0, 1)$ (and in particular, $h'_i = 1$) for r indices i with $1 \leq i \leq n$. Of these, at most $(r - 1) - (n - j)$ come from indices i for which $1 \leq i < j$. There are $(n - j + 1)$ other indices i with $j \leq i \leq n$, which gives $(r - 1) - (n - j) + (n - j + 1) = r$; hence all indices i for which $h'_i = 1$ force the equalities $h_i = 0$, and $(h, h') = (0 \ldots 0, h'_1 \ldots h'_{j-1} 1 \ldots 1)$. Thus $(e, e') \in \partial\{(h, h')\}$, $(e, e') \in \partial S_r$, and the lemma is proved.

Lemma 2. *Let $m_1, \ldots, m_k$ be nonnegative integers. Then the number of integers of weight r, which are less than $2^{m_1} + \ldots + 2^{m_k}$, is $[m_1, \ldots, m_k]_r$.*

Proof of Lemma 2. Let S be the set of binary words of weight r which are less than $2^{m_1} + \ldots + 2^{m_k}$. Let S_i, for each $i \in \{1, \ldots, k\}$, be the set of binary words of S which have 1's in the $2^{m_1}, \ldots, 2^{m_{i-1}}$ positions and 0 in the 2^{m_i} position. It can be shown that

$$S = S_1 \cup \ldots \cup S_k, \tag{19}$$

and futhermore, that the union on the right-hand side of (19) is disjoint.

To show (19), let $x \in S$. Then x must differ from $2^{m_1} + \ldots + 2^{m_k}$ at some first point 2^{m_i} since $x < 2^{m_1} + \ldots + 2^{m_k}$; hence $x \in S_i$. Reverse containment is trivial since the sets $S_1, \ldots, S_k$ are all subsets of S. To show that the union on the right-hand side of (19) is disjoint, let i and j be distinct elements of $\{1, \ldots, k\}$ and suppose without loss of generality that $i > j$. Let x_i and x_j be arbitrary members of S_i and S_j, respectively. Then x_i has a 1 at the 2^{m_j} position, and x_j has a 0 at the 2^{m_j} position; hence $x_i \neq x_j$ and $S_i \cap S_j = \emptyset$ for all $i \neq j$.

By the above conditions, it follows that

$$|S| = |S_1| + \ldots + |S_k|.$$

Let $i \in \{1, \ldots k\}$ and consider $|S_i|$. This is the number of integers of weight r which are less than $2^{m_1} + \ldots + 2^{m_k}$, have 1's in the $2^{m_1}, \ldots, 2^{m_{i-1}}$ positions, and a 0 in the 2^{m_i} position. There are m_i other positions $2^{m_i}, \ldots, 2^0$ from which to choose $r - (i-1) = r - i + 1$ positions to place 1's to form a binary word of weight r, giving $|S_i| = \binom{m_i}{r-i+1}$; hence

$$|S| = \binom{m_1}{r} + \binom{m_2}{r-1} + \ldots + \binom{m_k}{r-k+1}.$$

Since this is equivalent to $[m_1, \ldots, m_k]_r$ by definition, the proof of Lemma 2 is complete.

Lemma 3. *Let S be the m first nonnegative integers of weight r. Then the number of integers of weight s in $r(S)$ is $m^{(s/r)}$, and $r(S)$ is the set of all nonnegative integers, which are less than $2^{m_1} + \ldots + 2^{m_k}$, if $m = [m_1, \ldots, m_k]_r$ is the r-canonical representation of m.*

Lindström executes the proof of this lemma in [4]. Recall that $r(S)$ is the set of binary words related to S. For simplicial complexes, the boundary of a face, or binary word, of the complex is obtained by changing a 1 in one position to a 0.

Hence, a binary word of weight $s < r$ is in $r(S)$ if there exists a binary word $j \in S$ such that $i \subset j$ where i and j are viewed as sets. A binary word of weight $s > r$ is in $r(S)$ if all binary words j of weight r which satisfy $j \subset i$ are contained in S, where i and j are viewed as sets.

Once it has been shown that $r(S)$ is the set of binary words that are less than $M = 2^{m_1} + \ldots + 2^{m_k}$, application of Lemma 2 shows that the number of binary words of weight s, which are less than $2^{m_1} + \ldots + 2^{m_k}$, is $[m_1, \ldots, m_k]_s$. Since $m = [m_1, \ldots, m_k]_r$, then $[m_1, \ldots, m_k]_s = m^{(s/r)}$ by definition, and Lemma 3 follows. It remains to show

$$r(S) = \{i < 2^{m_1} + \ldots + 2^{m_k} \colon i \text{ is a binary word}\}. \tag{20}$$

Let $M = 2^{m_1} + \ldots + 2^{m_k}$ For the forward inclusion of (20), let $i \in r(S)$, and let s be the weight of i. If $s < r$, since there is a binary word $j \in S$ such that $i \subset j$, $i < M$ follows.

If $s > r$, assume to the contrary that $i \geq M$. Then for some integers $n_1, \ldots, n_s$

$$i = 2^{n_1} + \ldots + 2^{n_s} \geq 2^{m_1} + \ldots + 2^{m_k} = M.$$

If $j = 2^{n_1} + \ldots + 2^{n_r}$, then j is a binary word of weight r which satisfies $j \subset i$; hence $j \in S$. Then $i \geq M$ and $j < M$ imply $n_1 = m_1, \ldots, n_r = m_r$, and $k > r$. But by the r-canonical representation of m, $k \leq r$ must hold - a contradiction. Thus $i < M$ and the forward inclusion of (20) holds.

To prove the reverse inclusion, let $i < M$ be a binary word of weight s. If $s < r$ then for some index $l < k$ these inequalities result:

$$2^{m_1} + \ldots + 2^{m_l} \leq i < 2^{m_1} + \ldots + 2^{m_{l+1}}, \quad \text{or} \quad i < 2^{m_1}.$$

In the first case $m_{l+1} \geq r - l \geq 1$ follows from the r-canonical representation of m. Since $(s-l)$ 1's of i occur in the right-most m_{l+1} positions $2^{m_{l+1}-1}, \ldots, 2^0$, there are enough remaining 0's in these positions to construct a j of weight r which satisfies $i \subset j$ and

$$2^{m_1} + \ldots + 2^{m_l} \leq j < 2^{m_1} + \ldots + 2^{m_{l+1}}.$$

Thus $j \in S$ and $i \in r(S)$ follows by definition. In the second case, by definition of r-canonical form, $m_1 \geq r$; there are at least r positions to the right of 2^{m_1}; constructing j by starting with i and then filling in $(r - s)$ other of these positions demonstrates a j which satisfies $i \subset j$ and $j < M$; hence, $i \in r(S)$ by definition.

If $s > r$, then all binary words j of weight r for which $j \subset i$ also satisfy $j < M$; thus these binary words of weight r are all in S, and $i \in r(S)$ by definition. This completes the proof of Lemma 3.

B Appendix: Sample Program for Computation

Below is the listing of a utility program (written in C) that computes various quantities relating to cubical and simplicial complexes. Its purpose is to compute upper semipowers, as in Kruskal's Theorem; lower semipowers, as in Lemma 5; optimal number of faces of cubical complexes, as in Theorem 2; and validation of f-vectors of cubical complexes, as in Theorem 3. This program was used to create Tables 1 and 2.

```
/*************************************************************/
/* Program:     complex.c                                    */
/* Description: a utility program written in C designed to   */
/*              illustrate computations that arise in the    */
/*              thesis, particularly those of theorems 2 and 3 */
/*************************************************************/
#include <stdio.h>
#include <malloc.h>
#define f_vector long int *

void run_g_sum();
long int g_sum(long int r, long int i, long int m);
void run_comb();
long int comb(long int num, long int den);
void run_lower_pspwr();
long int lower_pspwr(long int f_i, long int s, long int r);
long int get_lower_m_value(long int f_i, long int r, long int i);
void run_upper_pspwr();
long int upper_pspwr(long int m, long int s, long int r);
long int get_upper_n_value(long int f_i, long int r);
f_vector get_f_vector(long int f_dim);
void run_max_min();
void check_complex(f_vector f, long int f_dim);
void fill_f_vector(f_vector f,long int dim,char *f_string);
long int length_f_vector(char *f_string);
void print_f_vector(f_vector f, long int f_dim);
long int menu_choice(f_vector f,long int f_dim);

main()
/*************************************************************/
/* The various options of the program compute the following:   */
```

```
/* 1) values for the auxiliary function in (\ref{GSum1})       */
/*        defined by Kruskal                                   */
/* 2) combinations                                             */
/* 3) lower pseudopowers as in (22)                            */
/* 4) upper pseudopowers as in (16)                            */
/* 5) entering an f-vector using the definition                */
/* 6) checking values for Theorem 2                            */
/* 7) validating the converse of Theorem 3                     */
/**************************************************************/
{   char *f_string;
    f_vector f=NULL;
    long int f_dim;
    long int go=1;
    long int loop, hold;
    long int response=0;

    printf("\n");
    while (response!=-1)
    {   response = menu_choice(f,f_dim);
        printf("\n");
        if (response==1)
        {   printf("***running g-sums***\n");
            run_g_sum();                        }
        if (response==2)
        {   printf("***computing nCr***\n");
            run_comb();                         }
        if (response==3)
        {   printf("***running lower pseudopowers***\n");
            run_lower_pspwr();                  }
        if (response==4)
        {   printf("***running upper pseudopowers***\n");
            run_upper_pspwr();                  }
        if (response==5)
        {   printf("***enter f-vector***\n");
            printf("Enter the dimension of the f-vector ");
            printf("followed by the entries of the \nf_vector");
            printf(" in the format 'dim f_0 f_1 ");
            printf("f_2 ... f_dim' (or -1 to quit):\n");
            scanf("%ld",&hold);
            if (hold!=-1)
            {   if (f!=NULL)
                    free(f);
                f_dim = hold;
                f = get_f_vector(f_dim);
```

```
                for (loop=0;loop<=f_dim;loop++)
                {   scanf("%ld",&hold);
                    f[loop]=hold;               }}
            printf("\n");                       }
        if (response==6)
        {   printf("***showing max/mins***\n");
            run_max_min();                      }
        if (response==7)
        {   printf("***f-vector test***\n");
            if (f==NULL)
                printf("First enter an f-vector.\n\n");
            else
                check_complex(f,f_dim);         }}
    return 0;
}

void run_g_sum()
/* The input subroutine for computing lower pseudopowers       */
{   long int r,i,m,check;
    long int go=1;
    while(go==1){
        printf("Enter values for r, i, and m (or -1 to quit):\n");
        scanf("%ld",&r);
        if (r!=-1)
            scanf("%ld%ld",&i,&m);
        if (r==-1)
            go=0;
        else
            check = g_sum(r,i,m);
            printf("g(%ld,%ld,%ld) = %ld\n\n",r,i,m,check);    }
    printf("\n");
    return;
}

long int g_sum(long int r, long int i, long int m)
/* computation of the lower pseudopowers                         */
{   long int loop;
    long int hold = 0;
    for (loop=0; loop <=m; loop++)
        hold += comb(m,loop)*comb(loop+i-1,r);
    return hold;
}

void run_comb()
```

```
/* The input subroutine for computing combinations              */
{   long int n,r;
    long int go=1;
    while (go==1)
    {   printf("Enter values for n and r (or -1 to quit):\n");
        scanf("%ld",&n);
        if (n!=-1)
        {   scanf("%ld",&r);
            printf("%ld C %ld = %ld\n\n",n,r,comb(n,r));      }
        else
        {   printf("\n");
            go=0;                 }}
}

long int comb(long int num, long int den)
/* computation of the combinations                              */
{   long int loop;
    long int hold=1;
    if ((num<0)||(den<0)||(den > num))
        return 0;
    if ((num==0)&&(den==0))
        return 1;
    for (loop=1; loop<=den; loop++)
    {   hold = hold * (num-den+loop);
        hold = hold / (loop);          }
    return hold;
}

void run_lower_pspwr()
/* input loop for computation of lower pseudopowers             */
{   long int m=0,r,s;
    while (m!=-1)
    {   printf("Enter m, s, and r to compute m_(s/r) ");
        printf("or -1 to quit:\n");
        scanf("%ld",&m);
        if (m!=-1)
        {   scanf("%ld %ld",&s,&r);
            printf("m_(s/r) = %ld\n\n",lower_pspwr(m,s,r));   }
        else
            printf("\n");              }
}

long int lower_pspwr(long int f_i, long int s, long int r)
/* computation loop for the lower pseudopowers                  */
```

```
{   long int f_i_sub_rr=0;
    long int f_i_sub_sr=0;
    long int i=1;
    long int m_value;
    long int first_loop=1;
    while((g_sum(r,i,m_value)>0)||(first_loop==1))
    {   first_loop=0;
        m_value = get_lower_m_value(f_i-f_i_sub_rr,r,i);
        if (m_value != -1)
        {   f_i_sub_rr += g_sum(r,i,m_value);
            if (g_sum(r,i,m_value)>0)
                f_i_sub_sr += g_sum(s,i,m_value);
            i++;                            }}
    return f_i_sub_sr;
}

long int get_lower_m_value(long int f_i, long int r, long int i)
/* does the work of (20); returns the highest m_i for which m  */
/* is not exceeded                                             */
{   long int hold=-1;
    long int checkloop=0;
    while (g_sum(r,i,checkloop)<=f_i){
        hold = checkloop;
        checkloop++;                        }
    return hold;
}

void run_upper_pspwr()
/* input loop for computation of lower pseudopowers            */
{   long int m=0,r,s;
    while (m!=-1)
    {   printf("Enter m, s, and r to compute m^(s/r) ");
        printf("or -1 to quit:\n");
        scanf("%ld",&m);
        if (m!=-1)
        {   scanf("%ld %ld",&s,&r);
            if ((r==0)&&(s>r)&&(m>0))
            {   printf("Can't make any weight ");
                printf("%ld words from ",s);
                printf("weight 0 words.\n\n");              }
            else if ((r==0)&&(m>1))
            {   printf("There cannot be more than one ");
                printf("word of weight 0\n\n");             }
            else
```

```
                printf("m^(s/r) = %ld\n\n",upper_pspwr(m,s,r));}
        else
            printf("\n");              }
}

long int upper_pspwr(long int m, long int s, long int r)
/* computation loop for the lower pseudopowers                   */
{   long int m_sup_rr=0;
    long int m_sup_sr=0;
    long int i=0;
    long int n_value=0;
    while(m_sup_rr < m)
    {   n_value = get_upper_n_value(m-m_sup_rr,r-i);
        m_sup_rr += comb(n_value,r-i);
        m_sup_sr += comb(n_value,s-i);
        i++;                            }
    return m_sup_sr;
}

long int get_upper_n_value(long int f_i,long int r)
/* does the work of (15); returns the highest m_i for which m  */
/* is not exceeded                                             */
{   long int hold = r;
    long int checkloop = r+1;
    if (r==0)
        return 0;
    while (comb(checkloop,r)<=f_i)
    {   hold = checkloop;
        checkloop++;                    }
    return hold;
}

f_vector get_f_vector(long int f_dim)
/* stores a new f-vector, a tuple of length f_dim + 1          */
{   f_vector f;
    f = (f_vector) (malloc( (f_dim+1)*sizeof(long int)));
    return f;
}

void run_max_min()
/* verifies the computational results of Theorem 2              */
{   long int f_r=0,f_s,r,s;
    long int f_r_sub_rr;
    long int f_r_sub_sr;
```

```
    long int diff_sup_sr;
    while (f_r!=-1)
    {   printf("Enter f_r, r, s, to find the minimum(maximum)");
        printf(" number of faces\nof dimension s given f_r ");
        printf("faces of dimension r where s<r(s>r) or ");
        printf("-1 to exit:\n");
        scanf("%ld",&f_r);
        if (f_r!=-1)
        {   scanf("%ld %ld",&r,&s);
            f_r_sub_rr = lower_pspwr(f_r,r,r);
            f_r_sub_sr = lower_pspwr(f_r,s,r);
            diff_sup_sr = upper_pspwr(f_r-f_r_sub_rr,s,r);
            f_s = f_r_sub_sr + diff_sup_sr;

            printf("There must be at %ld ",f_s);
            if (s<r) printf("least"); else printf("most");
            printf(" faces of dimension %ld\n",s);
            printf("given %ld faces of dimension ",f_r);
            printf("%ld\n\n",r);        }
        else
            printf("\n");               }
}

void check_complex(f_vector f, long int f_dim)
/* illustrates the results of the converse of Theorem 3          */
{   long int i,j;
    long int is_complex = 1;
    long int max_hold, min_hold;
    long int f_i_sub_ii, f_i_sub_ji, diff_sup_ji;
    long int f_j_sub_jj, f_j_sub_ij, diff_sup_ij;
    print_f_vector(f,f_dim);
    for (i=0;i<f_dim;i++){
        for (j=i+1;j<=f_dim;j++){
            f_i_sub_ii=lower_pspwr(f[i],i,i);
            f_i_sub_ji=lower_pspwr(f[i],j,i);
            diff_sup_ji=upper_pspwr(f[i]-f_i_sub_ii,j,i);
            max_hold=f_i_sub_ji + diff_sup_ji;
            if (f[j] > max_hold)
            {   printf("given f[%ld]=%ld, ",i,f[i]);
                printf("f[%ld]=%ld is too large: ",j,f[j]);
                printf("must be <= %d\n",max_hold);
                is_complex=0;           }
            f_j_sub_jj=lower_pspwr(f[j],j,j);
            f_j_sub_ij=lower_pspwr(f[j],i,j);
```

```
            diff_sup_ij=upper_pspwr(f[j]-f_j_sub_jj,i,j);
            min_hold=f_j_sub_ij + diff_sup_ij;
            if (f[i] < min_hold)
            {   printf("given f[%ld]=%ld, ",j,f[j]);
                printf("f[%ld]=%ld is too small: ",i,f[i]);
                printf("must be >= %d\n",min_hold);
                is_complex=0;           }}}
    if (is_complex==1)
        printf("The f-vector could be that of a cubical complex.");
    else
        printf("The f-vector can't be that of a cubical complex.");
    printf("\n\n");
    return;
}

void print_f_vector(f_vector f, long int f_dim)
{   long int loop;
    printf("f = (");
    for (loop = 0; loop<f_dim; loop++)
        printf("%ld,",f[loop]);
    printf("%ld)\n",f[f_dim]);
}

long int menu_choice(f_vector f,long int f_dim)
{   int response=-2;
    if (f!=NULL)
    {   printf("The current f-vector is: ");
        print_f_vector(f,f_dim);        }
    else printf("There is no f-vector in storage.\n");
    printf("Choose from the following options:\n");
    printf(" 1) run g_sums for r,i, and m\n");
    printf(" 2) compute nCr\n");
    printf(" 3) run lower pseudopowers for m,r, and s\n");
    printf(" 4) run upper pseudopowers for m,r, and s\n");
    printf(" 5) enter an f-vector\n");
    printf(" 6) given f_r, see max/min for f_s\n");
    printf(" 7) test f_vector as a cubical complex\n");
    printf("-1) exit to system\n");
    scanf("%d",&response);
    while((response==0)||(response<-1)||(response>7))
    {   printf("*improper response: use 1 through 7 or -1:\n");
        scanf("%d",&response);          }
    return response;
}
```

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Sum I, 1996; Spring 1996; Virginia Tech Graduate Teaching Assistant
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In reverse chronological order by term, responsibilities were: teaching freshman linear algebra, grading for freshman calculus, teaching two sections of freshman pre-calculus, grading for senior level numerical analysis, and workng in the mathematics department tutoring lab.

8/93-12/94 Self-Employed Tutor

While completing a B.S. in math, I worked as a tutor for various subjects in mathematics and computer science, ranging from freshman to junior level courses.

6/93-7/93, DataFocus, Inc. Consultant
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I contributed to the development of an in-house relational database business system using PowerBuilder and Oracle on the Windows/PC platform.

6/91-9/91 Sparrow Consulting Consultant
6/90-7/90 NSF/MS State Univ. Physics Dept. Programmer

I developed improvements for an in-house data collection program (Kmax) using Macintosh Programmer's Workshop, C, and Pascal on the Macintosh System 6.0 and 7.0 platforms, including an application specific program for analyzing radiation data taken from the Long Duration Exposure Facility (LDEF).

Publications

A Kruskal-Katona Theorem for Cubical Complexes, Master's Thesis (1996), for the completion of the degree of Master of Science (Mathematics) at Virginia Polytechnic Institute and State University.

References

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