How to Play the One-Lie Rényi-Ulam Game

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Abstract. The one-lie Rényi-Ulam liar game is a 2-player perfect information zero sum game, lasting q rounds, on the set $[n] := \{1, \dots, n\}$. In each round Paul chooses a subset $A \subseteq [n]$ and Carole either assigns 1 lie to each element of A or to each element of $[n] \setminus A$. Paul wins the regular (resp. pathological) game if after q rounds there is at least one (resp. at most one) element with one or fewer lies. We exhibit a simple, unified, optimal strategy for Paul to follow in both games, and use this to determine which player can win for all q, n and for both games.

Key words. Rényi-Ulam game, pathological liar game, searching with lies

1. Introduction

The *Rényi-Ulam liar game* and its many variations have a long and beautiful history, which began in [7,6] and is surveyed in [5]. The players Paul and Carole play a q-round game on a set of n elements, $[n] := \{1, \dots, n\}$. Each round, Paul splits the set of elements by choosing a *question* set $A \subseteq [n]$; Carole then completes the round by choosing to assign one *lie* either to each of the elements of A, or to each of the elements of $[n] \setminus A$. A given element is removed from play if it accumulates more than k lies, for some predetermined k. In choosing the question set A, we may consider the game to be restricted to the *surviving* elements, which have at most k lies. The game starts with each element having no associated lies. If after q rounds at most one element survives, Paul wins the original game; otherwise Carole wins. The dual *pathological liar* game, in which Paul wins whenever at least one element survives has recently been explored in [3,2]. The original game with $k = 1$ was solved in [4], wherein is found a three-page algorithm for Paul's strategy. We give a substantial simplification which simultaneously solves both the original and pathological one-lie ($k = 1$) games.

We represent a game state as $(q, \mathbf{x})$, where $\mathbf{x} = (x_0, x_1)$, x_0 denotes the number of elements with no lies, and x_1 denotes the number of elements with one lie. We denote Paul's question A by $\mathbf{a} = (a_0, a_1)$, where A contains a_0 elements that currently have no lies and a_1 elements that currently have a lie. Carole may then choose the successor state

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for the game, between $(q-1, \mathbf{y}')$ and $(q-1, \mathbf{y}'')$, where $\mathbf{y}' = (a_0, a_1 - a_0 + x_0)$ (attaching a lie to elements of $[n] \setminus A$) and $\mathbf{y}'' = (x_0 - a_0, x_1 - a_1 + a_0)$ (attaching a lie to elements of A).

Following Berlekamp in [1], the weight function for q questions, $\mathrm{wt}_q(\mathbf{x}) = (q+1)x_0 + x_1$, satisfies the relation $\mathrm{wt}_q(\mathbf{x}) = \mathrm{wt}_{q-1}(\mathbf{y}') + \mathrm{wt}_{q-1}(\mathbf{y}'')$, regardless of A. In the original game, Paul wants to decrease the weight as fast as possible; in the pathological game, Paul wants to keep the weight as high as possible. Since Carole is adversarial, Paul can do no better than choosing questions where the weight will divide in half. Hence, with q questions remaining, Carole has a winning strategy in the original (resp. pathological) game if the weight is greater (resp. less) than 2^q. The converse is not true; since all states and weights must be integral, Paul might not be able to divide the weight in half and Carole would then be able to cross the 2^q threshold.

2. The Splitting Strategy

Let $(q, \mathbf{x})$ be a game state. We call it *Paul-favorable* if $\mathrm{wt}_q(\mathbf{x}) \le 2^q$ (in the original game), or $\mathrm{wt}_q(\mathbf{x}) \ge 2^q$ (in the pathological game). Carole has a winning strategy from any state that is not Paul-favorable, by simply choosing the higher-weight (in the original game) or lower-weight (in the pathological game) state for her turns.

For $(q, \mathbf{x})$, let the *splitting* question A be $\mathbf{a} = \begin{cases} (\frac{x_0}{2}, \lfloor \frac{x_1}{2} \rfloor), & 2|x_0, \\ (\frac{x_0+1}{2}, \lceil \frac{x_1-q+1}{2} \rceil), & 2\not| x_0. \end{cases}$

We will show that this is the optimal question for Paul to ask, although it may not be legal because the game rules require $\mathbf{0} \le \mathbf{a} \le \mathbf{x}$ (coordinate-wise). Call Paul-favorable state $(q, \mathbf{x})$ *splitting* if the splitting question is a legal question for Paul to ask. For technical reasons, let us also call $(5, (3, 2))$ splitting in the ordinary game. In this special case the splitting question is not legal, but Paul still has a winning question in $\mathbf{a} = (2, 0)$.

Lemma 1. *$(q, \mathbf{x})$ is splitting if and only if at least one of the following holds:*

1. x_0 is even, or
2. $x_1 > q - 3$, or
3. $x_0 - x_1 < \frac{\mathrm{wt}_q(\mathbf{x}) + (3-q)(q+2)}{q+1}$.

Proof. $\mathbf{x}$ is always splitting if x_0 is even; otherwise, $\mathbf{x}$ is splitting if and only if $x_1 - q + 1 > -2$, which gives condition (2). Condition (3) holds if and only if $x_0(q+1) - x_1(q+1) < x_0(q+1) + x_1 + (3-q)(q+2)$, which is equivalent to Condition (2) by an easy computation. □

In the ordinary game, we will assume henceforth that q is minimal to make the state $(q, \mathbf{x})$ Paul-favorable. If $(q, \mathbf{x})$ is splitting then $(q-1, \mathbf{x})$ is splitting; Paul plays the game as if there were fewer questions remaining, and will have leftover questions at the end.

Example 1. In the pathological game, consider $(4, \mathbf{x})$ for $\mathbf{x} = (3, 1)$. We see that $\mathrm{wt}_4(\mathbf{x}) = 16 \ge 2^4$, so $(4, \mathbf{x})$ is Paul-favorable. However, it is not splitting since $3 - 1 \ge 2 = \frac{\mathrm{wt}_4(\mathbf{x}) + (3-4)(4+2)}{4+1}$.

This shows that Paul cannot always win from all Paul-favorable states. However, Paul can always win from all splitting states by asking the splitting question repeatedly.

Theorem 1. *Let* $(q, \mathbf{x})$ *be splitting. Let* $(q-1, \mathbf{y})$ *be the state after the splitting question and Carole's response. Then* $\mathrm{wt}_{q-1}(\mathbf{y}) = \lfloor \mathrm{wt}_q(\mathbf{x})/2 \rfloor$ *or* $\lceil \mathrm{wt}_q(\mathbf{x})/2 \rceil$*, and the state* $(q-1, \mathbf{y})$ *must be splitting.*

Proof. If x_0 is even, then $\mathrm{wt}_{q-1}(\mathbf{y}') = q\frac{x_0}{2} + \frac{x_0}{2} + \lceil \frac{x_1}{2} \rceil = \lceil \frac{x_0(q+1)+x_1}{2} \rceil = \lceil \mathrm{wt}_q(\mathbf{x})/2 \rceil$, and $\mathrm{wt}_{q-1}(\mathbf{y}'') = q\frac{x_0}{2} + \frac{x_0}{2} + \lfloor \frac{x_1}{2} \rfloor = \lfloor \frac{x_0(q+1)+x_1}{2} \rfloor = \lfloor \mathrm{wt}_q(\mathbf{x})/2 \rfloor$. If x_0 is odd, then $\mathrm{wt}_{q-1}(\mathbf{y}') = q\frac{x_0+1}{2} + \frac{x_0-1}{2} + \lceil \frac{x_1-q+1}{2} \rceil = \lceil \frac{x_0(q+1)+x_1}{2} \rceil = \lceil \mathrm{wt}_q(\mathbf{x})/2 \rceil$, and $\mathrm{wt}_{q-1}(\mathbf{y}'') = q\frac{x_0-1}{2} + \frac{x_0+1}{2} + x_1 - \lceil \frac{x_1-q+1}{2} \rceil = \lfloor \frac{x_0(q+1)+x_1}{2} \rfloor = \lfloor \mathrm{wt}_q(\mathbf{x})/2 \rfloor$.

In the pathological game, because $(q, \mathbf{x})$ is Paul-favorable, $\mathrm{wt}_q(\mathbf{x}) \geq 2^q$ and hence $\mathrm{wt}_{q-1}(\mathbf{y}) \geq 2^{q-1}$. In the ordinary game, because we assume that q is the minimal Paul-favorable question, $\mathrm{wt}_{q-1}(\mathbf{x}) \geq 2^{q-1}+1$, and hence $\mathrm{wt}_{q-1}(\mathbf{y}) \geq \lfloor \mathrm{wt}_q(\mathbf{x})/2 \rfloor = \lfloor (\mathrm{wt}_{q-1}(\mathbf{x}) + x_0)/2 \rfloor \geq 2^{q-2}$.

To show that $\mathbf{y}$ is splitting, we will show that $y_0 - y_1 < \frac{\mathrm{wt}_{q-1}(\mathbf{y})+(4-q)(q+1)}{q}$. For the pathological game, $\mathrm{wt}_{q-1}(\mathbf{y}) \geq 2^{q-1}$ and for the original game, $\mathrm{wt}_{q-1}(\mathbf{y}) \geq 2^{q-2}$. Therefore $\frac{\mathrm{wt}_{q-1}(\mathbf{y})+(4-q)(q+1)}{q}$ is > 1 for all q (except in the original game for $q = 4, 5, 6$, when it is > 0).

We now calculate $y_0 - y_1$ after the splitting question. If x_0 is even, then $y_0 - y_1 = -\lfloor \frac{x_1}{2} \rfloor$; or $y_0 - y_1 = -\lceil \frac{x_1}{2} \rceil$; in either case $y_0 - y_1 \leq 0$. If x_0 is odd, then $y_0 - y_1 = -1 - x_1 + \lceil \frac{x_1-q+1}{2} \rceil = \lceil \frac{-x_1-q-1}{2} \rceil \leq 0$; or $y_0 - y_1 = 1 - \lceil \frac{x_1-q+1}{2} \rceil$. Because $(q, \mathbf{x})$ is splitting, $x_1 - q + 1 > -2$; hence $y_0 - y_1 \leq 1$.

Hence $(q-1, \mathbf{y})$ is splitting except possibly in the ordinary game when x_0 and y_0 are odd, $y_0 - y_1 = 1$, and $4 \leq q \leq 6$. Since $\mathrm{wt}_{q-1}(\mathbf{y}) = (q+1)y_0 - 1$, $(q-1, y)$ is splitting unless $1 \geq \frac{(q+1)y_0 - 1 + (4-q)(q+1)}{q}$ if and only if $y_0 \leq q - 3$. Thus we are only concerned about states $(5, (3, 2))$ and $(q, (1, 0))$. The former is splitting by definition; in the latter, Paul has won. □

We now apply this strategy to the original and pathological one-lie games.

Corollary 1. *The original one-lie game is a win for Paul if and only if:*

1. $n \leq 2^q/(q+1)$*, for* n *even, or*
2. $n \leq (2^q - q + 1)/(q+1)$*, for* n *odd.*

Proof. The initial state is $(q, \mathbf{x})$ for $\mathbf{x} = (n, 0)$. If n is even, then the initial state is either splitting or not Paul-favorable, depending on whether Condition (1) holds. If n is odd and (2) holds, then $(q, \mathbf{x})$ is not splitting; however Paul can ask $(\frac{n+1}{2}, 0)$; in which case the next state $(q-1, \mathbf{y})$ will have $\mathbf{y} = (\frac{n+1}{2}, \frac{n-1}{2})$ or $\mathbf{y} = (\frac{n-1}{2}, \frac{n+1}{2})$. We have $\mathrm{wt}_{q-1}(\mathbf{y}) \leq q\frac{n+1}{2} + \frac{n-1}{2} = \frac{(q+1)n+(q-1)}{2} \leq 2^{q-1}$, applying $\mathrm{wt}_q(\mathbf{x}) \leq 2^q - (q-1)$. Since $y_0 - y_1 \leq 1$, $(q-1, \mathbf{y})$ will be splitting. If n is odd and (2) fails, then regardless of Paul's question the next state will not be Paul-favorable. □

Corollary 2. *The pathological one-lie game is a win for Paul if and only if:*

1. $n \geq 2^q/(q+1)$*, for* n *even, or*
2. $n \geq (2^q + q - 1)/(q+1)$*, for* n *odd.*

Proof. The initial state is $(q, \mathbf{x})$ for $\mathbf{x} = (n, 0)$. If n is even, then the initial state is either splitting or not Paul-favorable, depending on whether Condition (1) holds. If n is odd and (2) holds, then $(q, \mathbf{x})$ is not splitting; however Paul can ask $(\frac{n+1}{2}, 0)$; in which case the next state $(q-1, \mathbf{y})$ will have $\mathbf{y} = (\frac{n+1}{2}, \frac{n-1}{2})$ or $\mathbf{y} = (\frac{n-1}{2}, \frac{n+1}{2})$. We have

$\mathrm{wt}_{q-1}(\mathbf{y}) \geq q\frac{n-1}{2} + \frac{n+1}{2} = \frac{(q+1)n+(1-q)}{2} \geq 2^{q-1}$, applying $\mathrm{wt}_q(\mathbf{x}) \geq 2^q + (q-1)$. Since $y_0 - y_1 \leq 1$, $(q-1, \mathbf{y})$ will be splitting. If n is odd and (2) fails, then regardless of Paul's question the next state will not be Paul-favorable. □

References

1. E. R. Berlekamp. Block coding for the binary symmetric channel with noiseless, delayless feedback. In *Error Correcting Codes (Proc. Sympos. Math. Res. Center, Madison, Wis., 1968)*, pages 61–88. John Wiley, New York, 1968.
2. Robert B. Ellis, V. Ponomarenko, and Catherine H. Yan. The Rényi-Ulam pathological liar game with a fixed number of lies. *J. Combin. Theory Ser. A*. To appear.
3. Robert B. Ellis and Catherine H. Yan. Ulam's pathological liar game with one half-lie. *Int. J. Math. Math. Sci.*, (29-32):1523–1532, 2004.
4. Andrzej Pelc. Solution of Ulam's problem on searching with a lie. *J. Combin. Theory Ser. A*, 44(1):129–140, 1987.
5. Andrzej Pelc. Searching games with errors—fifty years of coping with liars. *Theoret. Comput. Sci.*, 270(1-2):71–109, 2002.
6. Alfréd Rényi. On a problem in information theory. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 6:505–516 (1962), 1961.
7. S. M. Ulam. *Adventures of a mathematician.* Charles Scribner's Sons, New York, 1976.