MATH 148 Fall 2007 Lecture Plan Chain Rule, 30 November 2007

Let $u = g(x)$ be differentiable at $x = a$. Let $y = f(u)$ be differentiable at $u = b = g(a)$. Form the composition $y = H(x) = (f \circ g)(x) = f(g(x))$. Then $H(x)$ is differentiable at $x = a$ and

$$H'(a) = f'(b) \cdot g'(a) = f'(g(a)) \cdot g'(a)\,.$$

To prove the Chain Rule, compute

$$\frac{dy}{dx} = H'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}\,.$$

Since $u = g(x)$ is differentiable at $x = a$,

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a)\,,$$

so define a function $G(x)$ that measures the discrepancy between the difference quotient and the value of the derivative as

$$G(x) = \frac{g(x) - g(a)}{x - a} - g'(a)$$

when $x \neq a$ and set $G(a) = 0$. Note that

$$\lim_{x \to a} G(x) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} - g'(a) = g'(a) - g'(a) = 0\,,$$

so $G(x)$ is continuous at $x = a$ and $\lim_{x \to a} G(x) = G(a) = 0$.

In a similar fashion we may define a continuous function $F(u)$ that measures the discrepancy between the difference quotient for f at $u = b$ and the value of the derivative $f'(b)$ at $u = b$, as

$$F(u) = \frac{f(u) - f(b)}{u - b} - f'(b)$$

when $u \neq b$ and set $F(b) = 0$. Then $F(u)$ is continuous at $u = b$ and $\lim_{u \to b} F(u) = F(b) = 0$.

Now

$$\begin{aligned}
[G(x) + g'(a)] \cdot (x - a) &= G(x) \cdot (x - a) + g'(a) \cdot (x - a) \\
&= \left[\frac{g(x) - g(a)}{x - a} - g'(a) \right] \cdot (x - a) + g'(a) \cdot (x - a) \\
&= \frac{g(x) - g(a)}{x - a} \cdot (x - a) - g'(a) \cdot (x - a) + g'(a) \cdot (x - a) \\
&= g(x) - g(a)\,.
\end{aligned}$$

In short,

$$g(x) - g(a) = [G(x) + g'(a)] \cdot (x - a)\,.$$

Moreover, exactly the *same* computations allow us to conclude that

$$f(u) - f(b) = [F(u) + f'(b)] \cdot (u - b)\,.$$

In this later formula for the difference $f(u) - f(b)$ we replace u with $g(x)$ and we replace b with $g(a)$ to get

$$f(g(x)) - f(g(a)) = [F(g(x)) + f'(g(a))] \cdot (g(x) - g(a))\,.$$

Now we use the former formula for the difference $g(x) - g(a)$ to rewrite this result as

$$f(g(x)) - f(g(a)) = [F(g(x)) + f'(g(a))] \cdot [G(x) + g'(a)] \cdot (x - a)\,.$$

Dividing both sides by $x - a$ and expanding the product on the right hand side gives us

$$\begin{aligned}
\frac{f(g(x)) - f(g(a))}{x - a} &= f'(g(a)) \cdot g'(a) + \\
&\quad f'(g(a)) \cdot G(x) + F(g(x)) \cdot g'(a) + F(g(x)) \cdot G(x)\,.
\end{aligned}$$

In the equation

$$\begin{aligned}\frac{f(g(x)) - f(g(a))}{x - a} &= f'(g(a)) \cdot g'(a) + \\ &\quad f'(g(a)) \cdot G(x) + F(g(x)) \cdot g'(a) + F(g(x)) \cdot G(x)\,,\end{aligned}$$

the first term on the right is the constant $f'(g(a)) \cdot g'(a)$. This means that the limit as x approaches a of the first term on the right is $f'(g(a)) \cdot g'(a)$. We shall see that the limiting value of each of the other terms on the right is 0.

For the second term on the right we have

$$\lim_{x \to a} f'(g(a)) \cdot G(x) = f'(g(a)) \cdot \lim_{x \to a} G(x) = f'(g(a)) \cdot G(a) = f'(g(a)) \cdot 0 = 0\,.$$

For the last two terms on the right we note that both F and g are continuous. Taking the limit as x approaches a of the third term on the right yields

$$\lim_{x \to a} F(g(x)) \cdot g'(a) = g'(a) \cdot \lim_{x \to a} F(g(x)) = g'(a) \cdot F(g(a)) = g'(a) \cdot F(b) = g'(a) \cdot 0 = 0\,.$$

For the fourth (last) term on the right, we get

$$\lim_{x \to a} F(g(x)) \cdot G(x) = \lim_{x \to a} F(g(x)) \cdot \lim_{x \to a} G(x) = F(b) \cdot G(a) = 0 \cdot 0 = 0\,.$$

Therefore

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} = f'(g(a)) \cdot g'(a)\,,$$

completing the proof of the Chain Rule.

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